COCYCLE AND ORBIT EQUIVALENCE SUPERRIGIDITY FOR MALLEABLE ACTIONS OF w-RIGID GROUPS

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Abstract. We prove that if a countable discrete group Γ is *w-rigid*, i.e. it contains an infinite normal subgroup H with the relative property (T) (e.g. $\Gamma = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, or $\Gamma = H \times H'$ with H an infinite Kazhdan group and H' arbitrary), and $\mathcal{V}$ is a closed subgroup of the group of unitaries of a finite separable von Neumann algebra (e.g. $\mathcal{V}$ countable discrete, or separable compact), then any $\mathcal{V}$-valued measurable cocycle for a measure preserving action $\Gamma \curvearrowright X$ of Γ on a probability space (X, μ) which is weak mixing on H and *s-malleable* (e.g. the Bernoulli action $\Gamma \curvearrowright [0,1]^{\Gamma}$) is cohomologous to a group morphism of Γ into $\mathcal{V}$. We use the case $\mathcal{V}$ discrete of this result to prove that if in addition Γ has no non-trivial finite normal subgroups then any orbit equivalence between $\Gamma \curvearrowright X$ and a free ergodic measure preserving action of a countable group Λ is implemented by a conjugacy of the actions, with respect to some group isomorphism $\Gamma \simeq \Lambda$.

0. Introduction

There has recently been increasing interest in the study of measure preserving actions of groups on probability measure spaces up to *orbit equivalence* (*OE*), i.e. up to isomorphisms of probability spaces taking the orbits of one action onto the orbits of the other. While the early years of this subject concentrated on the amenable case, culminating with the striking result that all ergodic m.p. actions of all countable amenable groups are undistinguishable under orbit equivalence ([Dy], [OW], [CFW]), the focus is now on proving *OE rigidity* results showing that, for special classes of (non-amenable) group actions, OE of $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ is sufficient to insure isomorphism of groups $\Gamma \simeq \Lambda$, or even conjugacy of the actions. Ideally, one seeks to prove this under certain conditions on the "source" group-action $\Gamma \curvearrowright X$ but no condition at all (or very little) on the "target" action $\Lambda \curvearrowright Y$, a type of result labeled *OE superrigidity.*

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The first OE rigidity phenomena were discovered by Zimmer, who used his celebrated cocycle superrigidity theorem to prove that any free ergodic m.p. actions of the groups $\mathrm{SL}(n,\mathbb{Z})$, $n=2,3,...$, are orbit inequivalent for different n's ([Z1,2]). A parallel discovery in von Neumann algebra theory, due to Connes ([C3]), showed that II_1 factors from groups with property (T) of Kazhdan have rigid symmetry structure. Ideas from ([C3], [Z2]) were used to produce more OE rigidity results in ([Ge], [GeGo], [P8], etc), while rigidity aspects of free and Gromov's word-hyperbolic groups began to emerge in ([A1,2]). An important development came with the remarkable work of Gaboriau on cost and ℓ^2-cohomology invariants for OE relations, showing in particular that free m.p. actions of the free groups $\mathbb{F}_n$ are OE inequivalent, for different $n \geq 1$ ([G1,2]). At the same time OE superrigidity phenomena started to unveil in the work of Furman who proved the striking result that, more than just being rigid, actions of higher rank lattices such as $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{T}^n$, for $n \geq 3$ odd, are in fact *OE superrigid*, i.e. any orbit equivalence between such an action and an arbitrary free m.p. action of a discrete group Λ comes from a conjugacy ([Fu1,2]). A startling new set of OE rigidity results was then established by Monod and Shalom, for doubly ergodic actions of products of word-hyperbolic groups ([MoS1,2]). The latest in this line is a result in ([P3]), showing that if Γ is an infinite conjugacy class (ICC) Kazhdan group then any *s-malleable* mixing action $\Gamma \curvearrowright (X,\mu)$ (such as the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$) is OE superrigid. Unlike previous OE rigidity results, which are all obtained in a measure theoretic framework (albeit each using different techniques), in ([P3]) this is a consequence of a "purely" von Neumann algebra rigidity result, in which the conjugacy class of $\Gamma \curvearrowright X$ is recovered from the *group measure space* von Neumann algebra $L^\infty X \rtimes \Gamma$, which apriori contains less information than the OE class of the action (cf [CJ]).

We prove in this paper an OE superrigidity result covering a much larger family of s-malleable actions $\Gamma \curvearrowright X$ than in ([P3]), with Γ merely required to contain an infinite normal subgroup with the relative property (T) (i.e. Γ is *w-rigid*) and the action assumed weak (rather than strong) mixing. We in fact prove a stronger form of OE superrigidity for the actions $\Gamma \curvearrowright X$, showing that if $\Lambda \curvearrowright Y$ is an arbitrary free m.p. action and $\Delta : X \simeq Y$ is an isomorphism of probability spaces taking each Γ-orbit into a Λ-orbit (not necessarily onto), then there exists a subgroup $\Lambda_0 \subset \Lambda$ such that the Γ and Λ_0 actions are conjugate, i.e. a suitable perturbation Δ_0 of Δ by an automorphism in the full group of Λ satisfies $\Delta_0 \Gamma \Delta_0^{-1} = \Lambda_0$.

The main result of this paper though is a *cocycle superrigidity* result for the actions $\Gamma \curvearrowright X$, from which the OE superrigidity is just a consequence. Thus, we show that any measurable cocycle for $\Gamma \curvearrowright X$ with values in an arbitrary discrete group Λ is equivalent to a group morphism of Γ into Λ. This result provides the first examples of *cocycle superrigid* group actions, i.e. actions having ALL cocycles with values into ANY discrete group cohomologous to group morphisms. The proof is very similar to (4.2 in [P1]), which served as a model for ([P2,3], [PSa]) as well. We use a von Neu-

mann algebra framework, as this is particularly suitable for the "deformation/rigidity" arguments involved. In fact, this setting allows us to prove that given any closed subgroup $\mathcal{V}$ of the unitary group of a finite von Neumann algebra, all $\mathcal{V}$-valued cocycles for $\Gamma \curvearrowright X$ can be untwisted. We use the same framework to show that if a group action is cocycle superrigid then it is OE superrigid. Partial cases of this general "principle" have been known for some time (see 4.2.9, 4.2.11 in [Z2], 3.3 in [Fu2], 2.4 in [Fu3]).

By using cocycles to study orbit equivalence of actions, we adopt in this paper an approach pioneered by Zimmer in the late 70's and which has been used, in one form or another, in many OE rigidity results obtained so far ([Z2], [Ge], [GeGo], [Fu1,2,3], [MoS1,2]). The past effectiveness of this method strongly motivated us to seek a suitable cocycle superrigidity result behind the OE superrigidity phenomena for malleable actions found in [P3]. In this respect, cocycle superrigidity with discrete targets is best suited for OE rigidity applications. In fact, as explained above, it can be viewed as a direct generalization (and strengthening) of OE superrigidity. Zimmer's cocycle superrigidity ([Z2]), in turn, is for linear algebraic groups as targets, thus extending Margulis' superrigidity but making it non-trivial to apply towards OE superrigidity. It was only after developing a series of new techniques that Furman could take full advantage of it to prove OE superrigidity of higher rank lattices ([Fu1,2,3]).

To state in more details the results in this paper, we need to recall some definitions and terminology (also used in [P2,3,4]). Thus, an inclusion of groups $H \subset \Gamma$ has the *relative property* (T) of Kazhdan-Margulis ([M], [K]) if any unitary representation of Γ that almost contains the trivial representation of Γ must contain the trivial representation of H. We also call such H a *rigid subgroup* of Γ. We'll consider two "weak normality" conditions for infinite subgroups $H \subset \Gamma$. Thus, H is *w-normal* (resp. *wq-normal*) in Γ if there exists a sequence of consecutive normal inclusions $H_i \subset H_{i+1}$ (resp. with H_{i+1} generated by elements $g \in \Gamma$ with $|gH_ig^{-1} \cap H_i| = \infty$) reaching from H to Γ (see 5.1 for the precise definitions). Infinite property (T) groups and the groups $\Gamma_0 \ltimes \mathbb{Z}^2$ with $\Gamma_0 \subset SL(2,\mathbb{Z})$ non-amenable have infinite normal rigid subgroups by ([K], [B]), and so are groups of the form $\Gamma_0 \ltimes \mathbb{Z}^n$, with Γ_0 arithmetic lattice in a classical Lie group and suitable n ([Va], [Fe]). If $H \subset \Gamma$ is wq-normal rigid then $H \subset (\Gamma * \Gamma_0) \times \Gamma_1$ is wq-normal rigid, $\forall \Gamma_1$ infinite. Both properties are closed to normal extensions.

Roughly, an action $\Gamma \curvearrowright^{\sigma} X$ is *s-malleable* if the flip automorphism on $X \times X$ is in the connected component of the identity in the centralizer of the double action $\sigma_g \times \sigma_g, g \in \Gamma$ (an additional "symmetry" requirement is in fact needed, see 4.3 for the exact definition). A typical example of s-malleable action is the *Bernoulli* Γ-*action* on $(X, \mu) = \Pi_{g \in \Gamma}(X_0, \mu_0)_g$, with $(X_0, \mu_0)_g$ identical copies of a standard probability space (X_0, μ_0), given by $\sigma_g((t_h)_h) = (t_{g^{-1}h})_h$, $(t_h)_h \in X$. More generally, if Γ acts on a countable set K and $(X, \mu) = \Pi_{k \in K}(X_0, \mu_0)_k$ with Γ acting on $(t_k)_k \in X$ by $\sigma_g((t_k)_k) = (t_{g^{-1}k})_k$, then σ is called a *generalized Bernoulli* Γ-*action* and it is still s-malleable. Note that Bernoulli actions are always (strongly) mixing, while a generalized

Bernoulli action is weak mixing iff $|\Gamma k| = \infty, \forall k$.

If σ is a Γ-action on (X, μ) and $\mathcal{V}$ a Polish group, then a (right) $\mathcal{V}$-valued measurable *cocycle* for σ is a measurable map $w : X \times \Gamma \to \mathcal{V}$ satisfying for each $g_1, g_2 \in \Gamma$ the identity $w(t, g_1)w(g_1^{-1}t, g_2) = w(t, g_1g_2)$, $\forall t \in X$ (a.e.). (N.B. All results below hold true, of course, for left cocycles as well. We choose to work with right measurable cocycles in this paper because in the von Neumann algebra framework, which is used for the proofs, they become left cocycles.) Two cocycles w, w' for σ are *cohomologous* (or *equivalent*) if there exists a measurable map $u : X \to \mathcal{V}$ such that for each $g \in \Gamma$ one has $w'(t, g) = u(t)^{-1}w(t, g)u(g^{-1}t)$, $\forall t \in X$ (a.e.). Note that a cocycle w is independent of the X variable iff it is a group morphism of Γ into $\mathcal{V}$.

A Polish group $\mathcal{V}$ is *of finite type* if it is isomorphic to a closed subgroup of the group of unitary elements $\mathcal{U}(N)$ of a countably generated finite von Neumann algebra N, equivalently a von Neumann algebra N having a faithful normal trace state τ such that N is separable in the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}, x \in N$. Countable discrete groups and separable compact groups are of finite type, as they can be embedded as closed subgroups of their group von Neumann algebra ([MvN1,2]). However, the only connected locally compact groups of finite type are the groups $\mathcal{V} = K \times V$ with K compact and $V \simeq \mathbb{R}^n$ a vector group (cf. [KaSi], [vNS]).

0.1. Theorem (Cocycle superrigidity). *Let $\Gamma \curvearrowright^{\sigma} X$ be a s-malleable action (e.g. a generalized Bernoulli Γ-action) and assume Γ has an infinite rigid subgroup H such that either H is wq-normal with σ mixing, or that H is w-normal with $\sigma_{|H}$ weak mixing. Let $\mathcal{V}$ be a Polish group of finite type. Then any $\mathcal{V}$-valued cocycle for σ is cohomologous to a group morphism of Γ into $\mathcal{V}$. More generally, if σ' is an action of the form $\sigma'_g = \sigma_g \times \rho_g \in \mathrm{Aut}(X \times Y, \mu \times \nu), g \in \Gamma$, where ρ is an arbitrary Γ-action on a standard probability space (Y, ν), then any $\mathcal{V}$-valued cocycle w for σ' is cohomologous to a $\mathcal{V}$-valued cocycle w' which is independent on the X-variable (i.e. w' comes from a cocycle of ρ).*

0.2. Corollary. *Let Γ be a discrete group having infinite wq-normal rigid subgroups and $\mathcal{V}$ a Polish group of finite type. Then any $\mathcal{V}$-valued cocycle for a Bernoulli Γ-action is cohomologous to a group morphism of Γ into $\mathcal{V}$.*

For the next result we denote by $\mathcal{R}_\theta$ the equivalence relation given by the orbits of a m.p. action $\Lambda \curvearrowright^{\theta} Y$ of a countable group Λ on a probability space (Y, ν). More generally, if $Y_0 \subset Y$ is a subset of positive measure then $\mathcal{R}_\theta^{Y_0}$ denotes the equivalence relation on Y_0 given by the intersection of the orbits of θ and the set Y_0. If θ is free ergodic then this is easily seen to only depend on $\mu(Y_0)$ ([Dy]), up to *isomorphism* of equivalence relations, i.e. up to isomorphism of probability spaces taking the orbits of one relation onto the orbits of the other (a.e.). As a consequence, it follows that if $t > 0$ and we take $m \geq t$, $\tilde{\theta}$ the product action of $\tilde{\Lambda} = \Lambda \times \mathbb{Z}/m\mathbb{Z}$ on the product probability space $\tilde{Y} = Y \times \mathbb{Z}/m\mathbb{Z}$ and $Y_0 \subset \tilde{Y}$ a subset of (product) measure t/m, then

the isomorphism class of $\mathcal{R}_{\tilde{\theta}}^{Y_0}$ only depends on t, not on the choice of m and $Y_0 \subset \tilde{Y}$. We call it the *amplification of* $\mathcal{R}_\theta$ *by* t and denote it $\mathcal{R}_\theta^t$.

Unlike Theorem 0.1, where the Γ-action σ doesn't need to be free, in the OE rigidity results below we have to assume freeness. If σ is a generalized Bernoulli action coming from an action of Γ on a set K as before, then the condition $|\{k \in K \mid gk \neq k\}| = \infty$, $\forall g \in \Gamma \setminus \{e\}$, insures that σ is free.

0.3. Theorem (OE superrigidity). *Let* $\Gamma \curvearrowright^\sigma X$ *be as in* 0.1. *and assume in addition that* Γ *has no nontrivial finite normal subgroups and* σ *is free. Let* θ *be an arbitrary free ergodic measure preserving action of a countable discrete group* Λ *on a standard probability space* (Y, ν). *If* Δ *is an isomorphism of probability spaces which takes* $\mathcal{R}_\sigma$ *onto* $\mathcal{R}_\theta^t$, *for some* $t > 0$, *then* $n = t^{-1}$ *is an integer and there exist a subgroup* $\Lambda_0 \subset \Lambda$ *of index* $[\Lambda : \Lambda_0] = n$, *a subset* $Y_0 \subset Y$ *of measure* $\nu(Y_0) = 1/n$ *fixed by* $\theta_{|\Lambda_0}$, *an inner automorphism* $\alpha \in \mathrm{Inn}(\mathcal{R}_\theta)$ *and a group isomorphism* $\delta : \Gamma \simeq \Lambda_0$ *such that* $\alpha \circ \Delta$ *takes* X *onto* Y_0 *and conjugates the actions* $\sigma, \theta_0 \circ \delta$, *where* θ_0 *denotes the action of* Λ_0 *on* Y_0 *implemented by* θ.

If moreover Γ *is infinite conjugacy class then any quotient of* $\Gamma \curvearrowright^\sigma X$ *is OE superrigid, i.e. any OE of* σ *and an arbitrary free action* $\Lambda \curvearrowright Y$ *comes from a conjugacy.*

0.4. Theorem (Superrigidity of embeddings). *Let* $\Gamma \curvearrowright^\sigma X$ *be as in* 0.1, $\Lambda \curvearrowright^\theta Y$ *an arbitrary free ergodic action and* $t > 0$.

If $\Delta : (X, \mu) \simeq (Y, \nu)^t$ *is an identification of* $\mathcal{R}_\sigma$ *with a subequivalence relation of* $\mathcal{R}_\theta^t$ *such that any* Γ*-invariant finite subequivalence relation of* $\mathcal{R}_\theta^t$ *must be contained in* $\mathcal{R}_\sigma$, *then* $t \leq 1$ *and there exists an isomorphism* $\delta : \Gamma \simeq \Lambda_0 \subset \Lambda$ *and* $\alpha \in \mathrm{Inn}(\mathcal{R}_\theta)$ *such that* $\alpha \circ \Delta$ *takes* X *onto a* Λ_0*-invariant subset* $Y_0 \subset Y$, *with* $\nu(Y_0) = t$, *and conjugates the actions* $\sigma, \theta_{|\Lambda_0}$ *with respect to the identification* $\delta : \Gamma \simeq \Lambda_0$.

An interesting application of 0.4 is as follows: Let μ_0 be a probability measure on $\{0, 1\}$ with unequal weights, i.e. $s = \mu_0(\{0\})/\mu_0(\{1\}) \neq 1$, let $(X, \mu) = (\{0, 1\}, \mu_0)^\Gamma$ and denote by $\mathcal{R}$ the countable equivalence relation on X generated by the Bernoulli Γ-action and the relation $\mathcal{R}_0$ given by $(t_g)_g \sim (t'_g)_g$ if $\exists F \subset \Gamma$ finite such that $t_g = t'_g, \forall g \in \Gamma \setminus F$, $\Pi_{g \in F} \mu_0(t_g) = \Pi_{g \in F} \mu_0(t'_g)$. If Γ is w-rigid and has no finite normal subgroups then $\mathscr{F}(\mathcal{R}) \supset s^{\mathbb{Z}}$ (see [P2]; equality is in fact shown for certain Γ, such as $\Gamma = SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$) and it is easy to see that any finite subequivalence relation of $\mathcal{R}$ invariant to the action of Γ on $\mathcal{R}$ given by $g(t, t') = (gt, gt')$ must be contained in $\mathcal{R}_\Gamma$. Thus, by 0.4 it follows that $\mathcal{R}^t$ cannot be implemented by a free action of a group, $\forall t > 0$. The first example of an equivalence relation with the property that all its amplifications are not implementable by free group actions was obtained in ([Fu2]).

The ideas behind the proofs of 0.1-0.4 (notably the deformation/rigidity arguments used in 0.1) are quintessentially "von Neumann algebra" in spirit. This made us favor a von Neumann algebra framework for the presentation, rather than a measure theoretical one. The two points of view are in fact equivalent, due to a well known observation

showing that if $\Gamma \curvearrowright^{\sigma} X$, $\Lambda \curvearrowright^{\theta} Y$ are free m.p. actions then an isomorphism of probability spaces $\Delta : X \simeq Y$ gives an OE of σ, θ if and only if, when regarded as an algebra isomorphism $\Delta : L^\infty X \simeq L^\infty Y$, Δ extends to an isomorphism of the von Neumann algebras $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ (cf. [Si], [Dy1,2], [FM]).

We summarize in Section 1 the tools from the theory of von Neumann algebras that we need in this paper, for convenience. In Section 2 we introduce the class of Polish groups of finite type and explain how measurable cocycles with values in such groups can be viewed as cocycles for actions on von Neumann algebras. Also, we discuss a *relative weak mixing* condition for actions on von Neumann algebras considered in ([P2]), which generalizes a concept introduced in the measure theoretic context by Furstenberg ([F]) and Zimmer ([Z3]) in the 1970's. It plays an important role in this paper. We show for instance that if $\Gamma \curvearrowright^{\sigma} X$ is a quotient of a cocycle superrigid action $\Gamma \curvearrowright^{\sigma'} X'$ and the latter is weak mixing relative to the former, then $\Gamma \curvearrowright X$ is cocycle superrigid as well. In Section 3 we prove a key criterion for "untwisting" cocycles with values in the unitary group of a finite von Neumann algebra N, extracted from proofs in ([P1,2]). It shows that a $\mathcal{U}(N)$-valued cocycle w for a weak mixing action $\Gamma \curvearrowright X$ is equivalent to a group morphism of Γ into $\mathcal{U}(N)$ iff the cocycles $w^l(t, s, g) = w(t, g)$ and $w^r(t, s, g) = w(s, g)$ for the double action $\sigma_g \times \sigma_g$ are equivalent. A similar statement holds true for arbitrary (non-commutative) finite von Neumann algebras. We also prove a hereditary result for weak mixing actions showing that in order to untwist a $\mathcal{V}$-valued cocycle, for some Polish group $\mathcal{V}$ of finite type, it is sufficient to untwist it as a cocycle with values in a larger Polish group of finite type.

In Section 4 we recall from ([P1-4]) the notion of *s-malleable* action and the proof that (generalized) Bernoulli actions have this property, then use a deformation/rigidity argument from ([P1,2]) to show that if a group Γ contains a large *rigid* part then any s-malleable Γ-action satisfies the criterion for untwisting cocycles from Section 3. In Section 5 we derive the cocycle superrigidity of s-malleable (e.g. Bernoulli) actions with values in Polish groups of finite type. We then prove a general principle showing that if $\Gamma \curvearrowright X, \Lambda \curvearrowright X$ have the same orbits, w denotes the associated cocycle and $v : X \to \Lambda$ implements an equivalence of w with a group morphism $\delta : \Gamma \to \Lambda$, then v gives rise to an automorphism α with graph in $\mathcal{R}_\sigma = \mathcal{R}_\theta$ which conjugates σ and $\theta \circ \delta$. In particular, this shows that cocycle superrigidity implies OE superrigidity, thus yielding the OE applications. Another general principle we prove is that any quotient of a cocycle superrigid weak mixing action of an ICC group is OE superrigid.

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1. Preliminaries

Although we are interested in actions of groups on the probability space, our approach will be functional analytical, using von Neumann algebra framework. This section is intended for readers who are less familiar with this field. Thus, we'll summarize here some basic tools in von Neumann algebras such as: the standard representation of a finite von Neumann algebra with a trace; the crossed product (resp. the group measure space) construction of a von Neumann algebra starting from an action of a discrete group on a finite von Neumann algebra (resp. on a probability space); orbit equivalence of actions as isomorphisms of group measure space von Neumann algebras; von Neumann subalgebras and the basic construction. Some knowledge in functional analysis and the spectral theorem should be sufficient to recover the omitted proofs.

1.1. Probability spaces as abelian von Neumann algebras. The "classical" measure theoretical approach is equivalent to a "non-classical" operator algebra approach due to a well known observation of von Neumann, showing that measure preserving isomorphisms between standard probability spaces (X, μ), (Y, ν) are in natural correspondence with *-algebra isomorphisms between their function algebras $L^\infty X$, $L^\infty Y$ preserving the functional given by the integral, $\tau_\mu = \int \cdot \mathrm{d}\mu$, $\tau_\nu = \int \cdot \mathrm{d}\nu$. More generally:

(1.1.1). Let $\Delta : (X, \mu) \to (Y, \nu)$ be a measurable map with $\nu \circ \Delta = \mu$. Then $\Delta^* : L^\infty Y \to L^\infty X$ defined by $\Delta^*(x)(t) = x(\Delta t), t \in X$, is an injective *-algebra morphism satisfying $\tau_\mu \circ \Delta^* = \tau_\nu$. Conversely, if $(X, \mu), (Y, \nu)$ are probability spaces and $\rho : L^\infty Y \to L^\infty X$ is an injective $*$-algebra morphism such that $\tau_\mu \circ \rho = \tau_\nu$, then there exists a measurable map $\Delta : X \to Y$ such that $\rho = \Delta^*$. Moreover, Δ is unique and onto, modulo a set of measure 0, and the correspondence $\Delta \mapsto \Delta^*$ is "contravariant" functorial, i.e. $(\Delta \circ \Delta')^* = \Delta'^* \circ \Delta^*$. Also, Δ is a.e. 1 to 1 if and only if Δ^* is onto and if this is the case then Δ^{-1} is also measurable and measure preserving.

There are two norms on $L^\infty X$ that are relevant for us in this paper, namely the ess-sup norm $\| \cdot \| = \| \cdot \|_\infty$ and the norm $\| \cdot \|_2$. Note that the unit ball $(L^\infty X)_1$ of $L^\infty X$ (in the norm $\| \cdot \|$) is complete in the norm $\| \cdot \|_2$. We will often identify $L^\infty X$ with the von Neumann algebra of (left) multiplication operators $L_x, x \in L^\infty X$, where $L_x(\xi) = x\xi$, $\xi \in L^2 X$. The identification $x \mapsto L_x$ is a *-algebra morphism, it is isometric (from $L^\infty X$ with the ess-sup norm into $\mathcal{B}(L^2 X)$ with the operatorial norm) and takes the $\| \cdot \|_2$-toplogy of $(L^\infty X)_1$ onto the strong operator topology on the image. Also, the integral $\tau_\mu(x)$ becomes the vector state $\langle L_x(1), 1 \rangle$, $x \in L^\infty X$. Moreover, if $\Delta : (X, \mu) \simeq (Y, \nu)$ for some other probability space (Y, ν), then $\rho = \Delta^{-1^*}$ extends to an (isometric) isomorphism of Hilbert spaces $L^2 X \simeq L^2 Y$ which conjugates the von Neumann algebras $L^\infty X \subset \mathcal{B}(L^2 X)$, $L^\infty Y \subset \mathcal{B}(L^2 Y)$ onto each other.

With this in mind, let us denote by $\mathrm{Aut}(X, \mu)$ the group of (classes modulo null sets of) *measure preserving automorphisms* $T : (X, \mu) \simeq (X, \mu)$ of the standard probability space (X, μ). Denote $\mathrm{Aut}(L^\infty X, \tau_\mu)$ the group of *-automorphisms of the von

Neumann algebra $L^\infty X$ that preserve the functional τ_μ, and identify $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(L^\infty X, \tau_\mu)$ via the map $T \mapsto (T^{-1})^*$. One immediate benefit of the functional analysis framework and of this identification is that it gives a natural Polish group topology on $\mathrm{Aut}(X,\mu)$, given by pointwise $\|\cdot\|_2$-convergence in $\mathrm{Aut}(L^\infty X, \tau_\mu)$, i.e. $\vartheta_n \to \vartheta$ in $\mathrm{Aut}(L^\infty X, \tau_\mu)$ if $\lim_n \|\vartheta_n(x) - \vartheta(x)\|_2 = 0$, $\forall x \in L^\infty X$.

An *action* of a discrete group Γ on the standard probability space (X,μ) is a group morphism $\sigma : \Gamma \to \mathrm{Aut}(X,\mu)$. Using the identification between $\mathrm{Aut}(X,\mu)$ and $\mathrm{Aut}(L^\infty X, \tau_\mu)$, we alternatively view σ as an action of Γ on $(L^\infty X, \tau_\mu)$, i.e as a group morphism $\sigma : \Gamma \to \mathrm{Aut}(L^\infty X, \tau_\mu)$. Although we use the same notation for both actions, the difference will be clear from the context. Furthermore, when viewing σ as an action on the probability space (X,μ), we'll use the simplified notation $\sigma_g(t) = gt$, for $g \in \Gamma, t \in X$. The relation between σ as an action on (X,μ) and respectively on $(L^\infty X, \tau_\mu)$ is then given by the equations $\sigma_g(x)(t) = x(g^{-1}t)$, $\forall t \in X$ (a.e.), which hold true for each $g \in \Gamma$, $x \in L^\infty X$.

The action σ is *free* if for any $g \in \Gamma, g \neq e$, the set $\{t \in X \mid gt = t\}$ has μ-measure 0. The action is *ergodic* if $X_0 \subset X$ measurable with $gX_0 = X_0$ (a.e.) for all $g \in \Gamma$, implies $X_0 = X$ or $X_0 = \emptyset$ (a.e.), in other words $\mu(X_0) = 0, 1$.

1.2. Finite von Neumann algebras. When viewed as an abelian von Neumann algebra with its integral functional, the natural generalization of a probability space is a von Neumann algebra N with a linear functional $\tau : N \to \mathbb{C}$ satisfying the conditions: $\tau(x^*x) \geq 0, \forall x \in N$ and $\tau(1) = 1$ (τ is a *state*); $\tau(x^*x) = 0$ iff $x = 0$ (τ is *faithful*); the unit ball $(N)_1 = \{x \in B \mid \|x\| \leq 1\}$ of N is complete in the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$ (τ is *normal*); $\tau(xy) = \tau(yx), \forall x, y \in N$ (τ is a *trace*). Such (N,τ) is called a *finite* von Neumann algebra (with its *trace* τ). We say that (N,τ) is *separable* if it is separable in the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}, x \in N$. The trace τ on a finite von Neumann algebra N is in general not unique, but there does exist a unique expectation of N onto its center $\mathcal{Z}(N)$, denoted Ctr_N and called the *central trace* on N, with the property that $Ctr_N(xy) = Ctr_N(yx), \forall x, y \in N$ ([D2]). Any (faithful normal) trace τ on N is the composition of Ctr_N with a (faithful normal) state on $\mathcal{Z}(N)$. In particular, if N is a *factor*, i.e. $\mathcal{Z}(N) = \mathbb{C}$, then N has a unique trace state $\tau = Ctr_N$, which is automatically normal and faithful.

If N is a finite von Neumann algebra and p, q are (selfadjoint) projections in N then $p \prec q$ (resp. $p \sim q$) in N, i.e. there exists a partial isometry $v \in N$ with $vv^* = p, v^*v \leq q$ (resp. $vv^* = p, v^*v = q$), if and only if $Ctr_N(p) \leq Ctr_N(q)$ (resp $Ctr_N(p) = Ctr_N(q)$). In particular, if $1 \sim q$ for some projection q in N then $q = 1$. By a celebrated theorem of Murray and von Neumann ([MvN1]), this purely algebraic condition is sufficient to ensure that N has a central trace, and thus a faithful normal trace state as well.

The representation of $L^\infty X$ as an algebra of left multiplication operators on L^2X generalizes to the non-commutative setting of finite von Neumann algebras (N,τ) as

follows: Denote by $L^2N = L^2(N,\tau)$ the completion of N in the norm $\|\cdot\|_2$. Then each element $x \in N$ defines an operator of left multiplication L_x on L^2N, by $L_x(\xi) = x\xi$, $\xi \in L^2N$. The map $N \ni x \mapsto L_x \in \mathcal{B}(L^2X)$ is clearly a *-algebra morphism. Due to the faithfulness of τ, it preserves the operatorial norm on N, with the normality of τ insuring that the image $L_N = \{L_x \mid x \in N\}$ is weakly closed in $\mathcal{B}(L^2X)$, i.e. L_N is a von Neumann algebra. It can be easily shown that the commutant L'_N of L_N in $\mathcal{B}(L^2N)$ is equal to the algebra R_N of operators of right multiplication by elements in N, and conversely $R'_N = L_N$. Also, by the definition, we have $\langle x1, 1\rangle = \tau(x)$. We will always identify N with its image $L_N \subset \mathcal{B}(L^2N)$ and call this the *standard representation* of (N,τ).

Note that if $(N,\tau) = (L^\infty X, \tau_\mu)$ then $L^2(N,\tau)$ coincides with the Hilbert space L^2X of square integrable functions on (X,μ). In this case $\xi \in L^2X$ can be viewed as the closed (densely defined) operator of (left) multiplication by ξ, whose spectral resolution lies in $L^\infty X$.

When (N,τ) is an arbitrary finite von Neumann algebra one can still interpret the elements in L^2N as closed linear operators on L^2N, as follows: For each $\xi \in L^2N$ let L^0_ξ be the linear operator with domain N (regarded as a vector subspace of L^2N) defined by $L_\xi(x) = \xi x$, $x \in N$. This operator extends to a unique closed operator L_ξ which commutes with R_N, equivalently its polar decomposition $L_\xi = u|L_\xi|$ has both the partial isometry u and the spectral resolution $\{e_s\}_{s>0}$ of $|L_\xi| = (L^*_\xi L_\xi)^{1/2}$ lying in N. Closed operators satisfying this property are said to be *affiliated with* N. In addition, L_ξ is *square integrable*, i.e. $\tau(L^*_\xi L_\xi) \overset{\text{def}}{=} \int s^2 \mathrm{d}\tau(e_s) = \|L_\xi(1)\|_2^2 = \|\xi\|_2^2 < \infty$. Noticing that $L_\xi(1) = \xi$, it follows that $\xi \mapsto L_\xi$ gives a 1 to 1 correspondence between L^2N and the space of square integrable operators affiliated with N. We will always view elements of L^2N in this manner.

If $\xi, \eta \in L^2N$ then their product (composition) as closed operators $\xi \cdot \eta$ is also a densely defined operator affiliated with N, i.e. it is of the form ub with u partial isometry in N and $b = |\xi\eta|$ a positive operator with spectral resolution $\{e_s\}_{s>0}$ in N and $\tau(b) \overset{\text{def}}{=} \int s\mathrm{d}\tau(e_s) < \infty$. We denote by L^1N the set of all such operators, which is easily seen to be a Banach space when endowed with the norm $\|ub\|_1 = \tau(b)$. Also, it has $L^2N \supset N$ as dense subspaces. Any $\zeta = \xi \cdot \eta^* \in L^1(M,\tau)$ defines a functional on M by $\tau(x\zeta) \overset{\text{def}}{=} \langle x\xi, \eta\rangle$, $x \in M$, which is positive iff $\zeta \geq 0$ as an operator. The norm of ζ as a functional on M coincides with $\|\zeta\|_1$ and in fact, as a space of functionals on M, L^1M is the predual of M, i.e. $(L^1M)^* = M$.

1.3. Actions of groups and crossed product algebras. Like in the commutative case, we denote by $\mathrm{Aut}(N,\tau)$ the group of automorphisms of the finite von Neumann algebra (N,τ) (i.e. the τ-preserving *-algebra isomorphisms of N onto N), and endow it with the Polish group topology given by point-wise $\|\cdot\|_2$-convergence. Note that any $\vartheta \in \mathrm{Aut}(N,\tau)$ preserves the Hilbert norm $\|\cdot\|_2$ and thus extends to a unitary operator

on L^2N which, if no confusion is possible, will still be denoted ϑ.

Given a discrete group Γ, an action σ of Γ on (N, τ) is a group morphism $\sigma : \Gamma \to \mathrm{Aut}(N, \tau)$. Since any σ_g extends to a unitary operator on L^2N, σ extends to a unitary representation of Γ on the Hilbert space L^2N, still denoted σ.

We use the (rather standard) notation N^σ to designate the *fixed point algebra* of the action, $N^\sigma = \{x \in N \mid \sigma_g(x) = x, \forall g \in \Gamma\}$.

A key tool in the study of actions is the *crossed product construction*, which associates to (σ, Γ) the von Neumann algebra $N \rtimes_\sigma \Gamma$ generated on the Hilbert space $\mathcal{H} = L^2N \overline{\otimes} \ell^2(\Gamma)$ by a copy of the algebra N, acting on $\mathcal{H}$ by left multiplication on L^2N, and a copy of the group Γ, acting on $\mathcal{H}$ as the operators $u_g = \sigma_g \otimes \lambda_g$, where $\sigma_g, g \in \Gamma$, is viewed here as unitary representation. In fact, $\{u_g\}_g$ is easily seen to be a multiple of left regular representation. In case $(N, \tau) = (L^\infty X, \mu)$ with σ coming from an action of Γ on (X, μ), this amounts to Murray-von Neumann's *group measure space construction* ([MvN1]).

The following more concrete description of $M = N \rtimes_\sigma \Gamma$ and its standard representation is quite useful: Identify $\mathcal{H} = \ell^2(\Gamma, L^2N)$ with the Hilbert space of ℓ^2-summable formal sums $\Sigma_g \xi_g u_g$, with "coefficients" ξ_g in L^2N and "indeterminates" $\{u_g\}_g$ labeled by the elements of the group Γ. Define a *-operation on $\mathcal{H}$ by $(\Sigma_g \xi_g u_g)^* = \Sigma_g \sigma_g(\xi^*_{g^{-1}}) u_g$ and let both N and the u_g's act on $\mathcal{H}$ by left multiplication, subject to the product rules $y(\xi u_g) = (y\xi)u_g$, $u_g(\xi u_h) = \sigma_g(\xi)u_{gh}, \forall g, h \in G$, $y \in N$, $\xi \in L^2N$. In fact, given any $\xi = \Sigma_g \xi_g u_g, \zeta = \Sigma_h \zeta_h u_h \in \mathcal{H}$ one can define the product $\xi \cdot \zeta$ as the formal sum $\Sigma_k \eta_k u_k$ with coefficients $\eta_k = \Sigma_g \xi_g \zeta_{g^{-1}k}$, the sum being absolutely convergent in the norm $\| \cdot \|_1$, with estimates $\|\eta_k\|_1 \leq \|\xi\|_2 \|\zeta\|_2$, $\forall k \in \Gamma$, by the Cauchy-Schwartz inequality.

$\xi \in \mathcal{H}$ is a *convolver* if $\xi\zeta \in \mathcal{H}$ (i.e. with the above notations $\eta_k \in L^2N$ and $\Sigma_k \|\eta_k\|_2^2 < \infty$) for all $\zeta \in \mathcal{H}$. M is then the algebra of all left multiplication operators $\zeta \mapsto \xi \cdot \zeta$ by convolvers ξ. Its commutant in $\mathcal{B}(\mathcal{H})$ is the algebra of all right multiplication operators $\zeta \mapsto \zeta\xi$ by convolvers ξ. If $T \in M$ then $\xi = T(1) \in \mathcal{H}$ is a convolver and T is the operator of left multiplication by ξ, with T^* corresponding to the left multiplication by ξ^*. The left multiplication by convolvers supported on $N = Nu_e$ gives rise to a multiple of the standard representation of N, while the left multiplication by the convolvers $\{u_g\}_g$ gives rise to a multiple of the left regular representation of Γ. The trace τ on N extends to all $N \rtimes_\sigma G$ by $\tau(\Sigma_g y_g u_g) = \tau(y_e) = \langle \xi, 1 \rangle = \langle \xi \cdot 1, 1 \rangle$, where $\xi = \Sigma_g y_g u_g$. The Hilbert space $\mathcal{H}$ naturally identifies with L^2M, while its subspace $M \subset L^2M$ identifies with the set of convolvers and the standard representation of M with the algebra of left multiplication by convolvers.

In case $N = L^\infty X$ and σ comes from an m.p. action $\Gamma \curvearrowright (X, \mu)$, the condition σ free is equivalent to $L^\infty X$ being maximal abelian in $M = L^\infty X \rtimes \Gamma$, i.e. if $x \in M$, $[x, L^\infty X] = 0$ then $x \in L^\infty X$. If σ is free, then M is a factor if and only if σ is ergodic. If σ is free and ergodic (equivalently $L^\infty X$ maximal abelian in M and M is a factor)

then there are two possibilities: Γ infinite, in which case M is a II_1 factor; $|\Gamma| = n < \infty$, in which case $M = M_{n\times n}(\mathbb{C})$ with the subalgebra $L^\infty X \subset M_{n\times n}(\mathbb{C})$ corresponding to a diagonal subalgebra of $M_{n\times n}(\mathbb{C})$.

1.4. Isomorphism of algebras from equivalence of actions. Let $\Gamma \curvearrowright^\sigma (X, \mu)$ and $\Lambda \curvearrowright^\theta (Y, \nu)$ be free m.p. actions of discrete groups on probability spaces. It is trivial to see that if $\Delta : (X, \mu) \simeq (Y, \nu)$ gives a *conjugacy* of σ, θ, i.e. $\Delta \circ \sigma_g = \theta_{\delta(g)} \circ \Delta$, $\forall g \in \Gamma$, for some group isomorphism $\delta : \Gamma \simeq \Lambda$, then $(\Delta^{-1})^* : L^\infty X \simeq L^\infty Y$ extends to an isomorphism of the group measure space algebras $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$, which takes a formal sum $\Sigma_g a_g u_g$ onto $\Sigma_g (\Delta^{-1})^*(a_g) v_{\delta(g)}$, where $u_g \in L^\infty X \rtimes \Gamma$, $v_h \in L^\infty Y \rtimes \Lambda$ are the canonical unitaries.

It has been shown by Singer ([Si]), Dye ([Dy1,2]) and Feldman-Moore ([FM]) that in fact much less than conjugacy is sufficient: If $\Delta : (X, \mu) \simeq (Y, \nu)$ is an isomorphism of probability spaces then $(\Delta^{-1})^* : L^\infty X \simeq L^\infty Y$ extends to an isomorphism of the corresponding group measure space von Neumann algebras if and only if Δ is an *orbit equivalence* (OE) of σ, θ, i.e. if for almost all $t \in X$ we have $\Delta(\Gamma t) = \Lambda\Delta(t)$.

This observation leads to the consideration of the *orbit equivalence relation* $\mathcal{R}_\sigma$ implemented on the probability space (X, μ) by the orbits of a free m.p. action $\Gamma \curvearrowright^\sigma (X, \mu)$, i.e. $(t, t') \in \mathcal{R}_\sigma$ if $\Gamma t = \Gamma t'$ ([Si], [FM]). More generally, if $X_0 \subset X$ is a measurable subset then one denotes by $\mathcal{R}_\sigma^{X_0}$ the equivalence relation on X_0 given by the intersection of the orbits of σ and the set X_0: $t, t' \in X_0$ are equivalent if $\Gamma t \cap X_0 = \Gamma t' \cap X_0$. If $\Lambda \curvearrowright^\theta (Y, \nu)$ is another free m.p. action and $Y_0 \subset Y$, $p_0 = \chi_{X_0}, q_0 = \chi_{Y_0}$, then an isomorphism of probability spaces $\Delta : (X_0, \mu_{X_0}) \simeq (Y_0, \nu_{Y_0})$ extends to a von Neumann algebra isomorphism $p_0(L^\infty X \rtimes \Gamma)p_0 \simeq q_0(L^\infty Y \rtimes \Lambda)q_0$ if and only if for almost all $t \in X$ we have $\Delta(\Gamma t \cap X_0) = \Lambda(\Delta(t)) \cap Y_0$, i.e. iff Δ takes the equivalence relation $\mathcal{R}_\sigma \cap (X_0 \times X_0)$ onto the equivalence relation $\mathcal{R}_\theta \cap (Y_0 \times Y_0)$.

To explain this result, it is convenient to consider a more general notion of equivalence relation (cf [FM]) and construct its associated von Neumann algebra. Thus, an equivalence relation $\mathcal{R}$ on X is called a *countable m.p. equivalence relation* if there exists a countable subgroup $\Gamma \subset \mathrm{Aut}(X, \mu)$ and a subset $N_0 \subset X$ of measure 0 such that for all $t \in X \setminus N_0$ the orbit of t under $\mathcal{R}$ coincides with Γt. The *full group* $[\mathcal{R}]$ of the equivalence relation $\mathcal{R}$ is the set of all $\phi \in \mathrm{Aut}(X, \mu)$ with the property that there exists a null set $N_0 \subset X$ such that the graph $G_\phi = \{(t, \phi(t)) \mid t \in X \setminus N_0\}$ is contained in $\mathcal{R}$. In this same spirit, if $\Gamma \subset \mathrm{Aut}(X, \mu)$ then $[\Gamma]$ denotes the group of automorphisms ϕ of (X, μ) which are locally implemented by elements on Γ, i.e. for which there exist a partition of X with measurable subsets $\{X_n\}_n$ and automorphisms $\phi_n \in \Gamma$ such that $\phi_{|X_n} = \phi_{n|X_n}, \forall n$. Note that if $\Gamma \subset \mathrm{Aut}(X, \mu)$ is a countable group, then Γ implements $\mathcal{R}$ iff $[\Gamma] = [\mathcal{R}]$.

Similarly, the *full pseudogroup* of a countable m.p. equivalence relation $\mathcal{R}$ (resp. of a subgroup $\Gamma \subset \mathrm{Aut}(X, \mu)$) is the set ${}_p[\mathcal{R}]$ (resp. ${}_p[\Gamma]$) of all measurable μ-m.p

isomorphisms $\phi : r(\phi) \simeq l(\phi)$, with $r(\phi), l(\phi) \subset X$ measurable and graph contained in $\mathcal{R}$ (resp. locally implemented by elements in Γ). This is easily seen to coincide with the set of all "local isomorphisms" of the form $\phi_{|Y_0}$ with $\phi \in {}_p[\mathcal{R}]$ (resp $\phi \in {}_p[\Gamma]$) and $Y_0 \subset X$. This set is endowed with a product given by $\phi\psi(t) \overset{\text{def}}{=} \phi(\psi(t))$, $t \in r(\phi\psi) \overset{\text{def}}{=} \{t \in r(\psi) \mid \psi(t) \in r(\phi)\}$ and inverse given by inverse of functions.

If $\mathcal{G}$ is a full pseudogroup on (X, μ) (coming from either a countable equivalence relation or a countable subgroup of $\text{Aut}(X, \mu)$) then its associated von Neumann algebra $L(\mathcal{G})$ is defined as follows: For each $\phi \in \mathcal{G}$, $a \in L^\infty X$ let $\phi(a) \in L^\infty X$ be defined by $\phi(a)(t) = a(\phi^{-1}(t))$, if $t \in l(\phi)$, $\phi(a)(t) = 0$ if $t \notin l(\phi)$. Denote $L_0(\mathcal{G})$ the algebra of formal finite sums $\Sigma_\phi a_\phi u_\phi$, with $a_\phi \in (L^\infty X) r(\phi)$ and "indeterminates" u_ϕ, product rule given by $(a_\phi u_\phi)(a_\psi u_\psi) = a_\phi \phi(a_\psi) u_{\phi\psi}$ and $*$-operation given by $(a_\phi u_\phi)^* = \phi^{-1}(a_\phi) u_{\phi^{-1}}$. Define $\tau(a_\phi u_\phi) = \int a_\phi i(\phi) \mathrm{d}\mu$, where $i(\phi)$ is the characteristic function of the largest set on which ϕ acts as the identity, then extend τ to all $L_0(\mathcal{G})$ by linearity. Denote by $L^2(\mathcal{G})$ the Hilbert space obtained by completing $L_0(\mathcal{G})/I_\tau$ in the norm $\|x\|_2 = \tau(x^* x)^{1/2}$, where $I_\tau = \{x \mid \langle x, x \rangle = 0\}$. The $*$-algebra $L_0(\mathcal{G})$ acts on $L^2(\mathcal{G})$ by left multiplication and $L(\mathcal{G})$ is defined to be its weak closure. In case $\mathcal{G} = {}_p[\mathcal{R}]$ for a countable m.p. equivalence relation, we denote $L({}_p[\mathcal{R}])$ by $L(\mathcal{R})$ (in the sense of 1.5 below).

Note that the algebra of coefficients $L^\infty X$ is maximal abelian in $L(\mathcal{R})$ and that any unitary element $u \in L(\mathcal{R})$ which normalizes $L^\infty X$, i.e. $uL^\infty X u^* = L^\infty X$, is of the form $u = au_\phi$, where a is a unitary element in $L^\infty X$ and $\phi \in [\mathcal{R}]$. It is useful to notice that, by maximality, there exist $\phi_n \in {}_p[\mathcal{R}]$ such that $i(\phi_n^{-1}\phi_m) = 0$, $\forall n \neq m$, and $\forall \phi \in {}_p[\mathcal{R}]$, $\exists X_n^\phi$ partition of $r(\phi)$ such that $\phi(t) = \phi_n(t)$, $\forall t \in X_n^\phi$ (a.e.). Note that these conditions amount to saying that $\{u_{\phi_n}\}_n$ is an orthonormal basis of $L([\mathcal{R}])$ over $L^\infty X$ (in the sense of 1.5 below).

1.4.1. Definition. If σ is free and ergodic then $\mathcal{R}_\sigma^{X_0}$ only depends on $t = \mu(X_0)$, up to isomorphism of equivalence relations (see e.g. [Dy1]). More generally, let $t > 0$ then choose an integer $n \geq t$ and denote by $\tilde{\sigma}$ the action of $\Gamma \times (\mathbb{Z}/n\mathbb{Z})$ on $\tilde{X} = X \times \mathbb{Z}/n\mathbb{Z}$ given by the product of σ and the left translations by elements in $\mathbb{Z}/n\mathbb{Z}$. Let $X_0 \subset \tilde{X}$ be a subset of measure t/n. It is then trivial to see that, up to isomorphism of equivalence relations, $\mathcal{R}_{\tilde{\sigma}}^{X_0}$ only depends on t (not on the choice of n and X_0). We denote the isomorphism class of $\mathcal{R}_{\tilde{\sigma}}^{X_0}$ by $\mathcal{R}_\sigma^t$ and call it the *amplification of* $\mathcal{R}_\sigma$ *by* t.

Let now $\mathcal{S}$ be another countable m.p. equivalence relation on the probability space (Y, ν) and let $\Delta : (X, \mu) \simeq (Y, \nu)$ be an isomorphism of probability spaces. It is now trivial from the definitions that Δ takes almost every orbit of $\mathcal{R}$ onto an orbit of $\mathcal{S}$ iff Δ conjugates the full groups $[\mathcal{R}], [\mathcal{S}]$, and also iff it conjugates the corresponding full pseudogroups. From the definition of $L(\mathcal{R})$, this later condition is clearly equivalent to the fact that $(\Delta^{-1})^* : L^\infty X \simeq L^\infty Y$ extends to a von Neumann algebra isomorphism of $L(\mathcal{R})$ onto $L(\mathcal{S})$. Such Δ is called an *orbit equivalence* (*OE*) of $\mathcal{R}, \mathcal{S}$. More generally:

1.4.2. Definition. Let $\Delta : (X, \mu) \to (Y, \nu)$ be a measurable measure preserving map. Δ is a *m.p. morphism* of $\mathcal{R}$ into $\mathcal{S}$ if there exists $N_0 \subset X_0$ with $\mu(N_0) = 0$ such that for all $t \in X \setminus N_0$, Δ takes the $\mathcal{R}$-orbits of t into the $\mathcal{S}$-orbits of $\Delta(t)$. Δ is called an *embedding* of $\mathcal{R}$ into $\mathcal{S}$ if it is an isomorphism of X onto Y and takes almost every orbit orbit of $\mathcal{R}$ into an orbit of $\mathcal{S}$. Δ is a *local OE* (or *local isomorphism*) of $\mathcal{R}$, $\mathcal{S}$ if there exists $N_0 \subset X$, $\mu(N_0) = 0$, such that $\forall t \in X \setminus N_0$, Δ is a bijection between the $\mathcal{R}$-orbit of t and the $\mathcal{S}$-orbit of $\Delta(t)$.

By (1.1.1), since any morphism Δ is surjective (a.e.), i.e. $\nu(Y \setminus \Delta(X)) = 0$, $\Delta^* : L^\infty Y \to L^\infty X$ is a faithful integral preserving von Neumann algebra embedding. It is trivial from the definitions that if Δ is an isomorphism of probability spaces then Δ is an embedding on $\mathcal{R}$ into $\mathcal{S}$ iff $\Delta[\mathcal{R}]\Delta^{-1} \subset [\mathcal{S}]$ and iff $(\Delta^{-1})^* : L^\infty X \simeq L^\infty Y$ extends to a von Neumann algebra isomorphism of $L(\mathcal{R})$ into $L(\mathcal{S})$.

If $\Delta : (X, \mu) \to (Y, \nu)$ is a local OE of $\mathcal{R}, \mathcal{S}$, then for each $\psi \in {}_p[\mathcal{S}]$ let $\Delta^*(\psi)$ be the *pull back* of ψ, i.e. the local isomorphism from $\Delta^{-1}(r(\psi))$ onto $\Delta^{-1}(l(\psi))$ which takes t onto the unique element $t' \in \mathcal{R}t$ with $\Delta(t') = \psi(\Delta(t))$. Thus, $\Delta^*(\psi) \in {}_p[\mathcal{R}]$ satisfies $\Delta \circ \Delta^*(\psi) = \psi \circ \Delta$. Another way of describing the pull back map is as follows: Let $\phi_n \in [\mathcal{R}]$ be an orthonormal basis of $\mathcal{R}$ and denote $X_n^\psi = \{t \in \Delta^{-1}(l(\psi)) \mid \psi^{-1}(\Delta(t)) = \Delta(\phi_n^{-1}(t))\}$; $\{X_n^\psi\}_n$ are then measurable, give a partition of $\Delta^{-1}(l(\psi))$ and we have $\psi^{-1}(\Delta(t)) = \phi_n^{-1}(t), \forall t \in X_n^\psi$. The pull back is clearly multiplicative (by the definitions), thus giving an embedding of full pseudogroups $\Delta^* : {}_p[\mathcal{S}] \to {}_p[\mathcal{R}]$. By the above definition of the von Neumann algebra associated to a full pseudogroup, this implies that Δ^* induces an isomorphism from $L(\mathcal{S})$ into $L(\mathcal{R})$, still denoted Δ^*. From the above remarks, if we denote v_ψ, $\psi \in {}_p[\mathcal{S}]$, the canonical partial isometry implementing ψ on $L^\infty Y$, i.e. $v_\psi a v_\psi{}^* = \psi(a)$, $a \in L^\infty Y$, and we let $p_n^\psi = \chi_{X_n^\psi} \in L^\infty X$, then with the above notations we trivially have:

1.4.3. Proposition. *$\Delta^*(v_\psi) = \Sigma_n p_n^\psi u_{\phi_n}$ defines a von Neumann algebra embedding of $L(\mathcal{S})$ into $L(\mathcal{R})$. Moreover, if we identify $L(\mathcal{S})$ with its image under Δ^*, then we have $(L^\infty Y)' \cap L(\mathcal{R}) = L^\infty X$, any element in $L(\mathcal{S})$ normalizing $L^\infty Y$ normalizes $L^\infty X$ and any orthonormal basis v_n of $L(\mathcal{S})$ over $L^\infty Y$ is an orthonormal basis of $L(\mathcal{R})$ over $L^\infty X$.*

1.5. von Neumann subalgebras and basic construction. If (Q, τ) is a finite von Neumann algebra then a *-subalgebra $N \subset Q$ closed in the weak operator topology (equivalently, $(N)_1$ closed in $\|\cdot\|_2$) and with the same unit as Q is called a *von Neumann subalgebra* of Q. For instance, if $Q = N \rtimes_\sigma \Gamma$ is the crossed product algebra corresponding to some actions σ of a discrete group Γ on the finite von Neumann algebra (N, τ) as in 1.3, then N identifies naturally with a von Neumann subalgebra of Q by viewing $a \in N$ as the "polynomial" au_e. Another important subalgebra of $N \rtimes \Gamma$ is the von Neumann subalgebra $L\Gamma$ generated by the canonical unitaries $\{u_g\}_g \subset L^\infty X \rtimes \Gamma$, i.e. the algebra of all convolvers $\Sigma_g c_g u_g$ with scalar coefficients $c_g \in \mathbb{C}$.

The restriction of functionals from Q to N implements a positive N-bimodular projection of $L^1(Q,\tau)$ onto $L^1(N,\tau_{|N})$ (Radon-Nykodim type theorem), whose restriction to Q gives the (unique) τ-preserving conditional expectation of Q onto N, denoted E_N. Restricted to L^2Q it implements the orthogonal projection of L^2Q onto L^2N, denoted e_N. Identifying $Q = L_Q \subset \mathcal{B}(L^2Q)$ gives $e_N x e_N = E_N(x)e_N, x \in Q$.

We denote by $\langle Q, e_N\rangle$ the von Neumann algebra generated in $\mathcal{B}(L^2Q)$ by $Q = L_Q$ and e_N. Since $e_N x e_N = E_N(x)e_N, \forall x \in Q$, and $\vee\{x(e_N(L^2Q)) \mid x \in Q\} = L^2Q$, it follows that span$Qe_NQ$ is a *-algebra with support equal to 1 in $\mathcal{B}(L^2Q)$ (i.e. if $p \in \mathcal{B}(L^2Q)$ is a projection with $pT = T$, $\forall T \in Qe_NQ$ then $p = 1$). Thus, $\langle Q, e_N\rangle = \overline{\text{sp}}^{\text{w}}\{xe_Ny \mid x, y \in Q\}$. Also $e_N\langle Q, e_N\rangle e_N = Ne_N$ implying that $\langle Q, e_N\rangle$ is a semifinite von Neumann algebra. This is called the (Jones) *basic construction* for the inclusion $N \subset Q$, with e_N its *Jones projection* ([J]).

We endow $\langle Q, e_N\rangle$ with a densely defined trace Tr by $Tr(\Sigma_i x_i e_N y_i) = \Sigma_i \tau(x_i y_i)$, for x_i, y_i finite sets of elements in Q. We denote by $L^2(\langle Q, e_N\rangle, Tr)$ the completion of spQe_NQ in the norm $\|x\|_{2,Tr} = Tr(x^*x)^{1/2}, x \in$ spQe_NQ. Exactly as in the case of finite von Neumann algebras with a trace, $\langle Q, e_N\rangle$ acts (as a von Neumann algebra) on $L^2(\langle Q, e_N\rangle, Tr)$ by left multiplication, and we call this the *standard representation* of $(\langle Q, e_N\rangle, Tr)$. Also, like in the finite case, the elements in $L^2(\langle Q, e_N\rangle, Tr)$ can be viewed as square summable (with respect to the semifinite trace Tr) operators affiliated with $\langle Q, e_N\rangle$, i.e. as closed operators $T \in \mathcal{B}(L^2(\langle Q, e_N\rangle, Tr))$ whose polar decomposition $T = u|T|$ has the partial isometry u and the spectral resolution $e_s, s > 0$, of $|T|$ lying in $\langle Q, e_N\rangle$ and satisfying $Tr(T^*T) \overset{\text{def}}{=} \int s^2 \text{d}Tr(e_s) < \infty$. The space $L^1(\langle Q, e_N\rangle, Tr)$ is defined similarly and for an operator T affiliated with $\langle Q, e_N\rangle$ we have $T \in L^2(\langle Q, e_N\rangle, Tr)$ iff $T^*T \in L^1(\langle Q, e_N\rangle, Tr)$, like in the finite case.

Any Hilbert subspace of L^2Q which is invariant under multiplication to the right by elements in N is a right Hilbert N-module. If $\mathcal{H} \subset L^2Q$ is a Hilbert subspace and f is the orthogonal projection onto $\mathcal{H}$ then $\mathcal{H}N = \mathcal{H}$ (i.e. $\mathcal{H}$ is a right N-module) iff f lies in $\langle Q, e_N\rangle$. An *orthonormal basis* over N for $\mathcal{H}$ is a subset $\{\eta_i\}_i \subset \mathcal{H}$ such that $\mathcal{H} = \overline{\Sigma_k \eta_k N}$ and $E_N(\eta_i^*\eta_{i'}) = \delta_{ii'}p_i \in \mathcal{P}(N), \forall i, i'$. Note that in this case we have $\xi = \Sigma_i \eta_i E_N(\eta_i^*\xi), \forall \xi \in \mathcal{H}$. A set $\{\eta_j\}_j \subset L^2Q$ is an orthonormal basis of $\mathcal{H}$ over N iff the orthogonal projection f of L^2Q on $\mathcal{H}$ satisfies $f = \Sigma_j \eta_j e_N \eta_j^*$, with $\eta_j e_N \eta_j^*$ projection $\forall j$. A simple maximality argument shows that any left Hilbert N-module $\mathcal{H} \subset L^2Q$ has an orthonormal basis (see [P6] for all this).

If $\xi \in L^2Q$ satisfies $E_N(\xi^*\xi) \in N$ (i.e. it is a bounded operator) then the closed operator $\xi e_N \xi^*$ lies in $\langle Q, e_N\rangle$ (i.e. it is bounded). If we denote by Φ the Q-bimodule map from sp$Qe_NQ \subset \langle Q, N\rangle$ into Q defined by $\Phi(xe_Ny) = xy, \forall x, y \in Q$, then $\tau \circ \Phi = Tr$ and Φ extends to a linear map from $L^1(\langle Q, e_N\rangle, Tr)$ onto L^1Q satisfying $\|\Phi(T)\|_1 \leq \|T\|_{Tr,1}$. If $\xi \in L^2Q$ then $\Phi(\xi e_N \xi^*) = \xi\xi^*$.

An action σ of a group Γ on (Q,τ) which leaves N invariant extends to a Tr-preserving action σ^N on $\langle Q, e_N\rangle$ by $\sigma_g^N(xe_Ny) = \sigma_g(x)e_N\sigma_g(y)$, $\forall x, y \in Q$, $g \in \Gamma$.

Moreover, since it preserves Tr, σ^N extends to a representation of Γ on the Hilbert space $L^2(\langle Q, e_N\rangle, Tr)$.

2. Some generalities on cocycles

2.1. Definition. Let σ be an action of a countable discrete group Γ on a standard probability space (X,μ) and $\mathcal{V}$ a Polish group. A (right) measurable 1-*cocycle* for σ with values in $\mathcal{V}$ is a measurable map $w : X\times\Gamma \to \mathcal{V}$ with the property that for all $g_1, g_2 \in \Gamma$ the equation

$$w(t,g_1)w({g_1}^{-1}t,g_2) = w(t,g_1g_2) \tag{2.1.1}$$

holds true μ-almost everywhere in $t\in X$. Two $\mathcal{V}$-valued cocycles w, w' are (measurably) *cohomologous* (or *equivalent*) if there exists $u : X\to\mathcal{V}$ measurable such that for all $g\in\Gamma$ we have

$$w'(t,g) = u(t)^{-1}w(t,g)u(g^{-1}t), \forall t \in X a.e. \tag{2.1.2}$$

We then write $w' \sim w$. We denote by $\mathrm{Z}^1(\sigma;\mathcal{V})$ the space of $\mathcal{V}$-valued cocycles for σ endowed with the topology of convergence in measure. Note that if $\mathcal{V}$ is abelian then plain multiplication of cocycles (as $\mathcal{V}$ valued functions) gives a group structure on $\mathrm{Z}^1(\sigma;\mathcal{V})$, which is clearly Polish with respect to the above topology.

Untwisting a cocycle w means showing it is equivalent to a cocycle which is independent on $t\in X$ (in the a.e. sense). It is immediate to see that such cocycles correspond precisely to group morphisms of Γ into $\mathcal{V}$: If $\delta:\Gamma\to\mathcal{V}$ is a group morphism and we define $w^\delta : \Gamma\times X\to\mathcal{V}$ by $w^\delta(t,g) = \delta(g), \forall t$, then w^δ is a cocycle; and conversely, if a cocycle $w : X\times\Gamma\to\mathcal{V}$ is so that for each $g\in\Gamma$ the map $t\mapsto w(t,g)$ is constant in $t\in X$ (a.e.), then there exists a unique group morphism $\delta:\Gamma\to\mathcal{V}$ such that $w = w^\delta$. We denote by $\mathrm{Z}^1_0(\Gamma;\mathcal{V})$ the subset of cocycles in $\mathrm{Z}^1(\Gamma;\mathcal{V})$ which are equivalent to group morphisms.

We only study in this paper cocycles with values in some closed subgroup of the group $\mathcal{U}(\mathcal{H})$ of unitaries acting on a separable Hilbert space $\mathcal{H}$, endowed with the usual Polish group structure given by the s^*-topology. Such cocycles are of particular interest because they can be interpreted in operator algebra framework, as elements of the von Neumann algebra of bounded measurable functions on X with values in $\mathcal{U}(\mathcal{H})$. Even more so, such cocycles will become (left) cocycles for a certain action of Γ on a von Neumann algebra. To explain this in details, we need some notations and general considerations.

Thus, if $(\mathcal{Y}, d_\mathcal{Y})$ is a separable complete metric space with finite diameter and (X,μ) a standard probability space, then we denote by $\mathcal{Y}^X$ the set of classes (modulo null sets) of measurable functions on X with values in $\mathcal{Y}$. We endow this set with the metric given

by the L^2-norm of the distance function, i.e. $d(f_1, f_2) = (\int d_{\mathcal{Y}}(f_1(t), f_2(t))^2 \mathrm{d}\mu(t))^{1/2}$, $f_1, f_2 \in \mathcal{Y}^X$). (N.B. Any other L^p-norm, $p \geq 1$, gives an equivalent metric.) $(\mathcal{Y}^X, d)$ is clearly a complete metric space, which is separable whenever $\mathcal{Y}$ is separable. This follows immediately by approximating elements in $\mathcal{Y}^X$ by step functions. Since such approximation will be used repeatedly in this paper, we mention it as a lemma. Its proof is standard and is thus omitted:

2.2. Lemma. *Let $(\mathcal{Y}, d_{\mathcal{Y}})$, $\mathcal{Y}^X$ be as above and $f_1, ..., f_m \in \mathcal{Y}^X$.*

1°. *For any $\varepsilon > 0$ there exist a finite partition $X_0, X_1, ..., X_n$ of X and elements $v_i^j \in \mathcal{Y}$, $1 \leq i \leq n, 1 \leq j \leq m$, such that $\mu(X_0) \leq \varepsilon$ and $d_{\mathcal{Y}}(f_j(t), v_i^j) \leq \varepsilon$, $\forall 1 \leq i \leq n, 1 \leq j \leq m$, $\forall t \in X_i$ a.e.*

2°. *If $Y \subset X$ is a set of positive measure and $(v_1, ..., v_n) \in \mathcal{Y}^n$ is an essential value of $(f_1, ..., f_n)$ on Y then there exists a decreasing sequence of subsets of positive measure $Y \supset Y_1 \supset ...$ such that $\mathrm{d}_{\mathcal{Y}}(f_j(t), v_j) \leq 2^{-n}$, $\forall t \in Y_n$, $n \geq 1$, $1 \leq j \leq m$.*

Let now $\mathcal{B}$ be a von Neumann algebra acting on the separable Hilbert space $\mathcal{H}$. Note that if $(\mathcal{B})_1$ denotes the unit ball of $\mathcal{B}$ (with respect to the operatorial norm $\|\cdot\|$ on $\mathcal{B}$) then the Borel structures on $(\mathcal{B})_1$ corresponding to the w, s and s^* topologies coincide (see e.g. [D1]). Thus, if we equip $(\mathcal{B})_1$ with either of these topologies and choose a Borel structure on X, then the Borel functions on X with values in $(\mathcal{B})_1$ will be the same. We denote by $L^\infty(X; \mathcal{B})$ the corresponding set of classes (modulo null sets) of (essentially) bounded μ-measurable functions on X with values in $\mathcal{B}$. It has a natural *-algebra structure given by point addition, multiplication and *-operation, and a C*-norm given by the ess-sup norm. This algebra acts in an obvious way on the Hilbert space $L^2(X; \mathcal{H})$ of square integrable functions on X with values in $\mathcal{H}$, as a von Neumann algebra.

There is a natural spatial isomorphism between the von Neumann algebra $L^\infty(X; \mathcal{B})$ acting this way on $L^2(X; \mathcal{H})$ and the tensor product von Neumann algebra $L^\infty(X, \mu)\overline{\otimes}\mathcal{B}$ acting on $L^2X\overline{\otimes}\mathcal{H}$, which sends a step function $f : X \to \mathcal{B}$ taking constant value $y_i \in \mathcal{B}$ on $Y_i \subset X$, for $\{Y_i\}_i$ a finite partition with measurable subsets of X, into the element $\Sigma_i \chi_{Y_i} \otimes y_i$.

Notice now that any embedding of a separable Polish group $\mathcal{V}$ as a closed subgroup of $\mathcal{U}(\mathcal{B})$ implements an embedding of $\mathcal{V}^X$ as a closed subgroup of the unitary group of the von Neumann algebra $L^\infty(X; \mathcal{B})$, and thus of the unitary group of $L^\infty X\overline{\otimes}\mathcal{B}$. By 2.2, when regarded as a subgroup of $\mathcal{U}(L^\infty X\overline{\otimes}\mathcal{B})$, $\mathcal{V}^X$ is the closure in the s^*-topology of the group of unitaries of the form $\Sigma_i \chi_{Y_i} \otimes v_i$, with $v_i \in \mathcal{V} \subset \mathcal{U}(\mathcal{B})$ and $\{Y_i\}_i$ finite partitions of X.

Notice also that if $\mathcal{V}$ is countable discrete, then any measurable map $f : X \to \mathcal{V}$ is given by a partition $\{Y_g\}_g$ of X into a countable family of measurable sets such that $f(t) = g, t \in Y_g, g \in \mathcal{V}$. Equivalently, when regarded as an element in $L^\infty X\overline{\otimes}\mathcal{B}$, f is of the form $f = \Sigma_g \chi_{Y_g} \otimes v_g^0$, where $\{v_g^0\}_g = \mathcal{V} \subset \mathcal{U}(\mathcal{B})$.

With (X,μ), $\mathcal{V} \subset \mathcal{U}(\mathcal{B})$ as above, let now $\sigma : \Gamma \to \mathrm{Aut}(X,\mu)$ be an action of a countable discrete group Γ on (X,μ). We still denote by $\sigma : \Gamma \to \mathrm{Aut}(L^\infty X, \tau_\mu)$ the action it implements on $L^\infty X$, as well as the $\mathcal{B}$-*amplification* of σ, i.e. the action of Γ on $L^\infty X \overline{\otimes} \mathcal{B}$ given by $\sigma_g \otimes id_{\mathcal{B}}$, $g \in \Gamma$. If $w : X \times \Gamma \to \mathcal{V}$ is a measurable map and we denote $w_g = w(\cdot, g) \in \mathcal{V}^X$, $g \in \Gamma$, with $\mathcal{V}^X$ viewed as a closed subgroup of the unitary group of $L^\infty(X;\mathcal{B}) = L^\infty X \overline{\otimes} \mathcal{B}$ as explained above, then conditions (2.1.1), (2.1.2) translate into properties of w_g as follows:

2.3. Lemma. *A measurable map $w : X \times \Gamma \to \mathcal{V}$ is a cocycle for the action σ of Γ on (X,μ) (i.e. it satisfies (2.1.1)) if and only if $w_g = w(\cdot, g) \in \mathcal{V}^X, g \in \Gamma$, satisfies with respect to the action $\sigma = \sigma \otimes id_{\mathcal{B}}$ of Γ on $L^\infty X \overline{\otimes} \mathcal{B}$ the relations*

$$w_g \sigma_g(w_h) = w_{gh}, \forall g, h \in \Gamma \tag{2.3.1}$$

Moreover, if $w, w' \in \mathrm{Z}^1(\sigma;\mathcal{V})$ and $u \in \mathcal{V}^X$ then u satisfies (2.1.2) if and only if, when viewed as an element in $\mathcal{U}(L^\infty X \overline{\otimes} \mathcal{B})$, it satisfies the relations

$$u^* w_g \sigma_g(u) = w'_g, \forall g \in \Gamma \tag{2.3.2}$$

Proof. Trivial by the definitions. □

2.4. Definition. Let Γ be a discrete group, $\mathcal{N}$ a von Neumann algebra and $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{N})$ an action of Γ on $\mathcal{N}$ (i.e. a group morphism of Γ into the group of automorphisms $\mathrm{Aut}(\mathcal{N})$ of the von Neumann algebra $\mathcal{N}$). A *cocycle* for σ is a map $w : \Gamma \to \mathcal{U}(\mathcal{N})$ satisfying equations (2.3.1) above. Two cocycles w, w' for σ are *cohomologous* (or *equivalent*) if there exists $u \in \mathcal{U}(\mathcal{N})$ such that (2.3.2) above holds true. More generally, a *local cocycle* for σ is a map w on Γ with values in the set of partial isometries of $\mathcal{N}$ satisfying $w_g \sigma_g(w_h) = w_{gh}, \forall g, h \in \Gamma$. It is easy to see that if w is such a local cocycle, then $p = w_e$ is a projection and for each $g \in \Gamma$ the element w_g belongs to the set $\mathcal{U}(p\mathcal{N}\sigma_g(p))$ of partial isometries with left support p and right support $\sigma_g(p)$.

With this terminology, Lemma 2.3 shows that a measurable map $w : X \times \Gamma \to \mathcal{U}(\mathcal{B})$ is a cocycle for an action σ of Γ on the probability space (X,μ) if and only if $w_g = w(\cdot, g), g \in \Gamma$, is a cocycle for the amplified action $\sigma_g \otimes id_{\mathcal{B}}$ of Γ on the algebra $L^\infty(X;\mathcal{B}) = L^\infty X \overline{\otimes} \mathcal{B}$. Also, equivalence of measurable cocycles w, w' for σ corresponds to their equivalence as algebra cocycles for $\sigma \otimes id_{\mathcal{B}}$.

In the rest of the paper, we will in fact concentrate on measurable cocycles with values in the following class of Polish groups:

2.5. Definition. A Polish group $\mathcal{V}$ is of *finite type* if it is isomorphic (as a Polish group) to a closed subgroup of the group of unitary elements of a separable (equivalently countably generated) finite von Neumann algebra. We denote by $\mathscr{U}_{fin}$ the class of all such groups.

2.6. Lemma. *Let $\mathcal{V}$ be a Polish group. The following conditions are equivalent:*

(i). $\mathcal{V} \in \mathscr{U}_{fin}$.

(ii). $\mathcal{V}$ is isomorphic to a closed subgroup of the group of unitary elements $\mathcal{U}(Q)$ of a separable type II_1 factor Q.

(iii). $\mathcal{V}$ is isomorphic to a closed subgroup $\mathcal{V} \subset \mathcal{U}(\mathcal{H})$ of the group of unitary operators on a separable Hilbert space $\mathcal{H}$ such that there exists $\xi \in \mathcal{H}$, $\xi \neq 0$, with the properties:

$$\text{(a)} \qquad \langle u_1 u_2 \xi, \xi \rangle = \langle u_2 u_1 \xi, \xi \rangle, \forall u_1, u_2 \in \mathcal{V},$$

$$\text{(b)} \qquad \overline{\mathrm{sp}} \mathcal{V}' \xi = \mathcal{H},$$

where $\mathcal{V}'$ denotes the commutant of $\mathcal{V}$ in $\mathcal{B}(\mathcal{H})$.

Proof. $(iii) \Rightarrow (i)$. If $\mathcal{V} \subset \mathcal{U}(\mathcal{H})$ is a closed subgroup and there exists a vector $\xi \in \mathcal{H}$ such that conditions (a), (b) are satisfied, then let $B = \mathcal{V}'' \subset \mathcal{B}(\mathcal{H})$ be the von Neumann algebra generated by $\mathcal{V}$ in $\mathcal{B}(\mathcal{H})$. By (a), $\tau(x) = \langle x\xi, \xi \rangle$, $x \in B$, is a normal trace on B which by (b) is faithful. Since the Hilbert space $\mathcal{H}$ on which it acts is separable, B follows finite and separable.

$(i) \Rightarrow (ii)$. If $\mathcal{V}$ is a closed subgroup of $\mathcal{U}(B)$ for a separable finite von Neumann algebra (B, τ), then let Q be the free product $Q = B * R$, with R the hyperfinite II_1 factor. By ([P7]) Q is a (separable) II_1 factor and since $\mathcal{V}$ is closed in $\mathcal{U}(B)$, it is closed in $\mathcal{U}(Q) \supset \mathcal{U}(B)$ as well.

$(ii) \Rightarrow (iii)$. If $\mathcal{V} \subset \mathcal{U}(Q)$ for a separable II_1 factor (Q, τ) and $\mathcal{H} = L^2(Q, \tau)$ is the standard representation of Q, then $\xi = 1$ is a trace vector for $\mathcal{U}(Q)$, thus for $\mathcal{V} \subset \mathcal{U}(Q)$ as well, and one has $\overline{\mathrm{sp}} \mathcal{V}' \xi \supset \overline{\mathrm{sp}} Q' \xi = \mathcal{H}$, showing that both conditions $(a), (b)$ of (i) are satisfied. $\square$

The next result provides some easy examples:

2.7. Lemma. 1°. *If a group $\mathcal{V}$ is either countable discrete or separable compact, then $\mathcal{V} \in \mathscr{U}_{fin}$.*

2°. *If $\mathcal{V}_n \in \mathscr{U}_{fin}, n \geq 1$, then $\Pi_{n \geq 1} \mathcal{V}_n \in \mathscr{U}_{fin}$.*

3°. *If $\mathcal{V} \in \mathscr{U}_{fin}$ and (X, μ) is a standard probability space then $\mathcal{V}^X \in \mathscr{U}_{fin}$.*

Proof. 1°. In both cases ($\mathcal{V}$ discrete, or $\mathcal{V}$ separable compact), the group von Neumann algebra $L\mathcal{V}$ of $\mathcal{V}$ is finite and has a normal faithful trace state (see e.g. [D]). Since $\mathcal{V}$ embeds into the unitary group of $L\mathcal{V}$ as the (closed) group of canonical unitaries $\{u_g\}_{g \in \mathcal{V}}$, it is of finite type by 2.6.

2°. If $\mathcal{V}_n$ is a closed subgroup in $\mathcal{U}(B_n)$, for some finite von Neumann algebra B_n with normal faithful trace state τ_n, $n \geq 1$, and we let (B, τ) be given by $B = \oplus_n B_n$,

$\tau(\oplus_n x_n) = \Sigma_n \tau(x_n)/2^n$, then (B, τ) is finite with normal faithful trace and $\Pi_n \mathcal{V}_n$ embeds into $\mathcal{U}(B)$ as the closed subgroup $\Pi_n \mathcal{V}_n$.

Part 3° is trivial, since $\mathcal{V}$ closed in $\mathcal{U}(B)$ implies $\mathcal{V}^X$ closed in the unitary group of the (separable) finite von Neumann algebra $L^\infty(X, \mu)\overline{\otimes}B$. $\square$

2.8. Notation. If σ is an action of a discrete group Γ on a finite von Neumann algebra (Q, τ) then we denote by $\mathrm{Z}^1(\sigma)$ the set of cocycles for σ. Note that if $N \subset Q$ is a von Neumann algebra such that $\sigma_g(N) = N, \forall g \in \Gamma$, and we denote $\rho_g = \sigma_{g|N}$ the restriction of σ to N, then we have a natural embedding $\mathrm{Z}^1(\rho) \subset \mathrm{Z}^1(\sigma)$. This embedding is in general not compatible with the equivalence of cocycles, i.e. two cocycles in $\mathrm{Z}^1(\rho)$ may be equivalent as cocycles in $\mathrm{Z}^1(\sigma)$ without being equivalent in $\mathrm{Z}^1(\rho)$. However, mixing properties of σ can entail the compatibility of the two equivalence relations. The following property from ([P2]) is quite relevant in this respect:

2.9. Definition. Let (Q, τ) be a finite von Neumann algebra and $N \subset Q$ a von Neumann subalgebra. Let $\sigma : \Gamma \to \mathrm{Aut}(Q, \tau)$ be an action of a discrete group Γ on (Q, τ) that leaves N globally invariant. The action σ is *weak mixing* (resp. *mixing*) *relative to* N if for any finite set $F \subset Q \ominus N$ and any $\varepsilon > 0$ there exists $g \in \Gamma$ (resp. there exists $K_0 \subset \Gamma$ finite) such that $\|E_N(\eta^* \sigma_g(\eta'))\|_2 \leq \varepsilon, \forall \eta, \eta' \in F$ (resp. $\forall g \in \Gamma \setminus K_0$).

It is easy to check that if $N \subset Q$ are abelian von Neumann algebras then 2.9 becomes the weak mixing property introduced in the 1970's by Furstenberg [F] and Zimmer [Z3]. (N.B. I am grateful to Alex Furman for drawing my attention on this work.)

To give alternative characterizations of property 2.9, recall some notations from 1.4. Thus, let $N \subset Q \subset \langle Q, e\rangle$ be the basic construction for $N \subset Q$, with $e = e_N$ denoting the Jones projection. Let Tr be the canonical trace on $\langle Q, e\rangle$ and σ^N the action of Γ on $(\langle Q, e\rangle, Tr)$ given by $\sigma_g^N(xey) = \sigma_g(x)e\sigma_g(y)$, $x, y \in Q$.

The next two results and their proofs are in the spirit of (Sec. 3 and 5.2 in [P2]).

2.10. Lemma. *The following conditions are equivalent:*

(i). σ is weak mixing relative to N.

(i'). For any finite subset $F \subset L^2Q \ominus L^2N$, with $E_N(\eta^*\eta)$ bounded $\forall \eta \in F$, and any $\varepsilon > 0$ there exists $g \in \Gamma$ such that $\|E_N(\eta^* \sigma_g(\eta'))\|_2 \leq \varepsilon, \forall \eta, \eta' \in F$.

(ii). There exist $g_n \to \infty$ in Γ and an orthonormal basis $\{1\} \cup \{\xi_j\}_{j \geq 1}$ of L^2Q over N such that $\lim_{n\to\infty} \|E_N^Q(\xi_i^* \sigma_{g_n}(\xi_j))\|_2 = 0$, $\forall i, j$.

(iii). Any $\xi \in L^2(\langle Q, e\rangle, Tr)$ fixed by σ^N lies in the subspace $L^2(e\langle Q, e\rangle e, Tr) = L^2(Ne)$.

Moreover, if $(Q, \tau) = (P\overline{\otimes}N, \tau_P \otimes \tau_N)$ and σ leaves both $N = 1 \otimes N$ and $P = P \otimes 1$ globally invariant, then σ is weak mixing (resp. mixing) relative to N if and only if $\sigma_{|P}$ is weak mixing (resp. mixing). Also, if $\sigma_{|P}$ is weak mixing then any fixed point of σ lies in N.

Proof. $(ii) \Rightarrow (iii)$. To prove that all fixed points of σ^N as an action on $L^2(\langle Q, e\rangle, Tr)$ are under e it is sufficient to prove that any projection $f \in \langle Q, e\rangle$ with $Tr(f) < \infty$ which is fixed by σ^N must satisfy $f \leq e$. Assume there exists a σ^N-invariant f with $fe \neq f$, equivalently $(1-e)f(1-e) \neq 0$. Since $(1-e)$ is σ^N-invariant, by replacing f by an appropriate spectral projection of $(1-e)f(1-e) \neq 0$ we may thus assume $0 \neq f \leq 1-e$.

Denote $f_n = \Sigma_{i=1}^n \xi_i e \xi_i^*$. Then $Tr(f_n) < \infty$, $\forall n$, and $e + f_n \nearrow 1$ in $\langle Q, e\rangle$. Thus $\|f_n f - f\|_{2,Tr} = \|(e+f_n)f - f\|_{2,Tr} \to 0$.

Let $\varepsilon > 0$ and n be large enough such that $\|f_n f - f\|_{2,Tr} < \varepsilon/3$. Since σ satisfies (ii), there exists $g \in \Gamma$ such that $\|E_N(\xi_j^* \sigma_g(\xi_i))\|_2 < \varepsilon/(3n), 1 \leq i, j \leq n$. We have $\sigma^N(f_n) = \Sigma_{i=1}^n \sigma_g(\xi_i) e \sigma_g(\xi_i^*)$ and thus

$$\|\sigma_g^N(f_n) f_n\|_{2,Tr}^2 = Tr(f_n \sigma_g^N(f_n)) = \Sigma_{i,j=1}^n Tr(\xi_j e \xi_j^* \sigma_g(\xi_i) e \sigma_g(\xi_i^*))$$

$$= \Sigma_{i,j=1}^n \|E_N(\xi_j^* \sigma_g(\xi_i))\|_2^2 \leq \varepsilon^2/9.$$

This further implies

$$Tr(f) = Tr(\sigma_g^N(f) f) = Tr(\sigma_g^N(f) f_n f) + Tr(\sigma_g^N(f)(f - f_n f))$$

$$\leq |Tr(\sigma_g^N(f - f f_n) f_n f)| + |Tr(\sigma_g^N(f f_n) f_n f)| + |Tr(\sigma_g^N(f)(f - f_n f)|$$

$$\leq \|f - f f_n\|_{2,Tr} \|f_n f\|_{2,Tr} + \|\sigma_g^N(f_n) f_n\|_{2,Tr} \|f \sigma_g^N(f)\|_{2,Tr} + \|f - f_n f\|_{2,Tr} \|\sigma_g^N(f)\|_{2,Tr}$$

$$\leq (\varepsilon/3 + \varepsilon/3 + \varepsilon/3)\|f\|_{2,Tr} = \varepsilon \|f\|_{2,Tr}.$$

Taking $\varepsilon < 1$, this shows that $f = 0$, a contradiction. Thus, $f \leq e$, finishing the proof of $(ii) \Rightarrow (iii)$.

$(i') \Rightarrow (i)$ and $(i') \Rightarrow (ii)$ are trivial. To prove $(i) \Rightarrow (i')$, note first that for $\eta \in L^2Q$ the condition $E_N(\eta^*\eta) \in N$ (i.e. bounded) is equivalent to $\eta e \eta^*$ bounded. Let $F \subset L^2Q \setminus L^2N$ be a finite set such that $E_N(\eta^*\eta) \in N$, $\forall \eta \in F$. By (i) there exists $g_n \in \Gamma$ such that $\lim_n \|E_N(x^* \sigma_{g_n}(y))\|_2 = 0$, for all x, y in the (separable) von Neumann algebra Q_0 generated by $\eta|\eta|^{-1}$ and by the spectral projections of $\eta\eta^*$ for $\eta \in F$. Showing that $\lim_n \|E_N(\xi^* \sigma_{g_n}(\eta))\|_2 = 0$, for some $\xi, \eta \in F$, is equivalent to showing that $\lim_n \|\xi e \xi^* \sigma_{g_n}^N(\eta e \eta^*)\|_{2,Tr} = 0$. But since for spectral projections p of $\xi\xi^*$ we have $\|p\xi e \xi^* p\| \leq \|\xi e \xi^*\|$, the latter is trivially implied by the fact that $\lim_n \|p \xi e \xi^* p \sigma_{g_n}^N(q \eta e \eta^* q)\|_{2,Tr} = 0$ for all $\xi, \eta \in F$ and p, q spectral projections of $\xi\xi^*, \eta\eta^*$ corresponding to finite intervals.

$(iii) \Rightarrow (i')$. Assume by contradiction that there exists a finite subset $F \subset L^2Q \ominus L^2N$, with $E_N(\eta^*\eta) \in N, \forall \eta \in F$, and $\varepsilon_0 > 0$ such that $\Sigma_{\eta,\eta'} \|E_N(\eta^* \sigma_g(\eta'))\|_2^2 \geq \varepsilon_0$, $\forall g \in \Gamma$. This is equivalent to the fact that $b = \Sigma_{\eta \in F} \eta e \eta^*$ satisfies

$$Tr(b \sigma_g^N(b)) \geq \varepsilon_0, \forall g \in \Gamma.$$

Note that $b \in L^2(\langle Q, e\rangle, Tr)$. Denote by $K = \overline{\text{co}}\{\sigma_g^N(b) \mid g \in \Gamma\} \subset L^2(\langle Q, e\rangle, Tr)$, the closure being in the weak topology on the Hilbert space. Let $b_0 \in K$ be the unique element of minimal norm $\|\cdot\|_{2,Tr}$. Noticing that $\sigma_g^N(K) = K$ and $\|\sigma_g^N(b_0)\|_{2,Tr} = \|b_0\|_{2,Tr}$, $\forall g$, by the uniqueness of b_0 it follows that $\sigma_g^N(b_0) = b_0$, $\forall g \in \Gamma$. Thus $b_0 = eb_0e$. But by the condition $F \perp L^2N$ it follows that $be = 0$, thus $0 = Tr(beb_0) = Tr(bb_0) \geq \varepsilon_0$, a contradiction.

To prove that last part of the statement, notice first that if $\sigma_{|P}$ is not weak mixing then it has an invariant finite dimensional subspace $\mathcal{H}_0 \subset L^2P \ominus \mathbb{C}$. If $\{\eta_i\}_i$ is an orthonormal basis of $\mathcal{H}_0$, then $\Sigma_i \eta_i \otimes N$ is σ-invariant. Equivalently, $f = \Sigma_i \eta_i e_N \eta_i^* \in \langle Q, e\rangle$ is fixed by σ^N. It is also finite and satisfies $fe = 0$, showing that σ is not weak mixing relative to N (by the equivalence of (i) and (iii)).

Conversely, if $\sigma_{|P}$ is weak mixing then any orthonormal basis $\{1\} \cup \{\xi_i\}$ of L^2P is an orthonormal basis of Q over N and condition (ii) above is clearly satisfied. By the equivalence of (ii) and (i), it follows that σ is weak mixing relative to N.

Now if $\sigma_{|P}$ is weak mixing and $x \in P\overline{\otimes}N$ is fixed by σ then xex^* is fixed by σ^N and, since σ is weak mixing relative to $N = 1 \otimes N$, by (iii) it follows that $xex^* \in Ne$, i.e. $x \in N$. $\square$.

Note that if $N = \mathbb{C}$ then Definition 2.8 amounts to σ being weak mixing and $(i) \Leftrightarrow (iii)$ amounts to the characterization of this property stating that the only invariant finite dimensional vector subspace of L^2N is $\mathbb{C}$.

2.11. Lemma. *Let σ, ρ be actions of the discrete group Γ on finite von Neumann algebras $(P, \tau), (N, \tau)$ and w a cocycle for the diagonal product action $\sigma_g \otimes \rho_g$, $g \in \Gamma$, on $(P\overline{\otimes}N, \tau)$. Let (P', τ') be a finite von Neumann algebra containing (P, τ) and σ' an extension of σ to an action of Γ on (P', τ') such that σ' is weak mixing relative to P. If $w_g(\sigma'_g \otimes \rho_g)(v) = vw'_g, \forall g \in \Gamma$, for some cocycle w' with values in $\mathcal{U}(N) \subset \mathcal{U}(P'\overline{\otimes}N)$ and some $v \in \mathcal{U}(P'\overline{\otimes}N)$, then $v \in P\overline{\otimes}N$, i.e. $w \sim w'$ as cocycles for $\sigma_g \otimes \rho_g, g \in \Gamma$. In particular, it follows that if $\Gamma \curvearrowright^{\sigma} (X, \mu)$ is the quotient of an action $\Gamma \curvearrowright^{\sigma'} (X', \mu')$ such that σ' is weak mixing relative to $L^\infty X$ and $\mathcal{V} \subset \mathcal{U}(N)$ is a closed subgroup, then any $\mathcal{V}$-valued cocycle w for σ which can be untwisted to a group morphism $w' : \Gamma \to \mathcal{V}$ as a cocycle for σ', with (un)twister $v : X' \to \mathcal{V}$, then v comes from a m.p. map $v : X \to \mathcal{V}$, and thus w can be untwisted as a cocycle for σ.*

Proof. If we consider the basic construction $P\overline{\otimes}N \subset P'\overline{\otimes}N \subset \langle P'\overline{\otimes}N, e\rangle$ and denote by $\lambda_g = \sigma'_g \otimes (\text{Ad}w'_g \circ \rho_g)$ then $\lambda_g(vP\overline{\otimes}N) = vP\overline{\otimes}N$, $\forall g$. But by 2.10, σ' weak mixing relative to P implies λ weak mixing relative to $P\overline{\otimes}N$, so by 2.10 again we have $v \in P\overline{\otimes}N$. $\square$

We end this section with a result showing that the equivalence class of a cocycle for an action of a group on a finite von Neumann algebra is closed in the topology of uniform $\|\cdot\|_2$-convergence. We also show that if two cocycles are cohomologous

then any partial isometry giving a "partial equivalence" can be extended to a unitary element that implements an actual equivalence.

2.12. Lemma. *Let w, w' be cocycles for the action σ of a group Γ on a finite von Neumann algebra (Q, τ).*

1°. *If $\|w_g - w'_g\|_2 \leq \delta, \forall g \in \Gamma$, then there exists a partial isometry $v \in Q$ such that $\|v - 1\|_2 \leq 4\delta^{1/2}$ and $w_g\sigma_g(v) = vw'_g$, $\forall g \in \Gamma$.*

2°. *If for any $\varepsilon > 0$ there exists $u \in \mathcal{U}(Q)$ such that $\|w_g\sigma_g(u) - uw'_g\|_2 \leq \varepsilon$, $\forall g \in \Gamma$, then w, w' are cohomologous.*

3°. *If w, w' are cohomologous and $v \in Q$ is a partial isometry satisfying $w_g\sigma_g(v) = vw'_g$, $\forall g \in \Gamma$, then there exists $u \in \mathcal{U}(Q)$ such that $uv^*v = v$ and $w_g\sigma_g(u) = uw'_g$, $\forall g \in \Gamma$.*

Proof. 1°. Note first that if we let $\pi_g(\xi) = w'_g\sigma_g(\xi)w_g^*$, for each $g \in \Gamma$, $\xi \in L^2Q$, then π_g are unitary elements and $g \mapsto \pi_g$ is a unitary representation of Γ on L^2Q. Indeed, this is an immediate consequence of the cocycle relation (2.4.1).

Let $K = \overline{\text{co}}^w\{w'_gw_g^* \mid g \in \Gamma\}$, the closure being in the weak topology on the Hilbert space L^2Q. Note that K is actually contained in Q, more precisely $\|y\| \leq 1$, $\forall y \in K$. In particular K is bounded in the norm $\|\cdot\|_2$, so it is compact in the weak topology. Also, since $\|w'_gw_g^* - 1\|_2 = \|w'_g - w_g\|_2$, $\forall g \in \Gamma$, by taking convex combinations and weak closure we get $\|y - 1\|_2 \leq \delta$, $\forall y \in K$. Let $y_0 \in K$ be the unique element of minimal norm-$\|\cdot\|_2$. Noticing that $\pi_g(K) = K$ and $\|\pi_g(\xi)\|_2 = \|\xi\|_2$, $\forall g \in \Gamma, \xi \in L^2Q$, by the uniqueness of y_0 it follows that $w'_g\sigma_g(y_0)w_g^* = \pi_g(y_0) = y_0, \forall g \in \Gamma$. Thus $w'_g\sigma_g(y_0) = y_0w_g, \forall g \in \Gamma$. In addition, $y_0 \in K$ implies $\|y_0 - 1\|_2 \leq \delta$. But then the partial isometry v in the polar decomposition of y_0 also satisfies $w'_g\sigma_g(v) = vw_g, \forall g \in \Gamma$, while by ([C2]) we have $\|v - 1\|_2 \leq 4\delta^{1/2}$.

2°. For the proof of this part and part 3° below, we use Connes' "2 by 2 matrix trick" ([C1]). Thus, let $\tilde{\sigma}$ be the action of Γ on $\tilde{Q} = M_{2\times 2}(Q) = Q \otimes M_{2\times 2}(\mathbb{C})$ given by $\tilde{\sigma}_g = \sigma_g \otimes id$. If $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ is a matrix unit for $M_{2\times 2}(\mathbb{C}) \subset \tilde{Q}$, then $\tilde{w}_g = w_ge_{11} + w'_ge_{22}$ is a cocycle for $\tilde{\sigma}$. If $B \subset \tilde{Q}$ denotes the fixed point algebra of the action $\text{Ad}\tilde{w}_g \circ \tilde{\sigma}$, then $e_{11}, e_{22} \in B$ and the existence of a unitary element $u \in Q$ intertwining w, w' is equivalent to the fact that e_{11}, e_{22} are equivalent projections in B. Moreover, any partial isometry $v \in Q$ satisfying $w_g\sigma_g(v) = vw'_g$, $\forall g \in \Gamma$, gives a partial isometry $v \otimes e_{12}$ in B with left, right supports given by $vv^* \otimes e_{11}$, $v^*v \otimes e_{22}$.

Thus, by part 1° and the hypothesis, there exist partial isometries $v_n \in B$ such that $v_nv_n^* \leq e_{11}, v_n^*v_n \leq e_{22}$ and $\|v_nv_n^* - e_{11}\|_2 = \|v_n^*v_n - e_{22}\|_2 \to 0$. But this implies e_{11}, e_{22} are equivalent in B (for instance, because they have the same central trace in B).

3°. Since w, w' are equivalent, the projections e_{11}, e_{22} are equivalent in B. On the other hand, if v satisfies the given condition, then $\tilde{v} = v \otimes e_{12} \in B$ and $\tilde{v}\tilde{v}^* \leq e_{11}, \tilde{v}^*\tilde{v} \leq$

e_{22}. This implies the projections $e_{11} - \tilde{v}\tilde{v}^*$ and $e_{22} - \tilde{v}^*\tilde{v}$ are equivalent in B, say by a partial isometry $\tilde{v}' = v' \otimes e_{12}$. Then $u = v' + v$ clearly satisfies the condition. □

3. Techniques for untwisting cocycles

Let σ be an action of a countable discrete group Γ on the standard probability space (X, μ). Denote by $\tilde{\sigma}$ the associated *double action* of Γ on $(X \times X, \mu \times \mu)$, given by the diagonal product $\tilde{\sigma}_g(t_1, t_2) = (gt_1, gt_2)$, $t_1, t_2 \in X$.

The main result in this section is a criterion for untwisting cocycles, extracted from proofs in ([P1]) and ([P2]).

3.1. Theorem. *Assume σ is weakly mixing and let ρ be another action of Γ on a standard probability space (Y, ν). Let $\mathcal{V} \in \mathscr{U}_{fin}$ and $w \in \mathrm{Z}^1(\sigma \times \rho; \mathcal{V})$, where $\sigma \times \rho$ is the diagonal product action on $(X \times Y, \mu \times \nu)$ given by $\sigma_g \times \rho_g, g \in \Gamma$. Denote by w^l, w^r the cocycles for the diagonal product action $\tilde{\sigma} \times \rho$ defined by $w^l(t_1, t_2, s, g) = w(t_1, s, g)$, $w^r(t_1, t_2, s, g) = w(t_2, s, g)$, $g \in \Gamma, t_1, t_2 \in X, s \in Y$. Assume there exists some separable finite von Neumann algebra (N, τ) such that $\mathcal{V}$ can be realized as a closed subgroup of $\mathcal{U}(N)$ and such that $w^l \sim w^r$ as $\mathcal{U}(N)$-valued cocycles for $\tilde{\sigma} \times \rho$. Then w is equivalent to a $\mathcal{V}$-valued cocycle which is independent on the X-variable.*

Conversely, if $w \in \mathrm{Z}^1(\sigma \times \rho; \mathcal{V})$ is equivalent to a cocycle in $\mathrm{Z}^1(\rho; \mathcal{V})$ then $w^l \sim w^r$ in $\mathrm{Z}^1(\tilde{\sigma} \times \rho; \mathcal{V})$.

In particular, if $w \in \mathrm{Z}^1(\sigma; \mathcal{V})$ then $w \in \mathrm{Z}^1_0(\sigma; \mathcal{V})$ if and only if $w^l \sim w^r$ in $\mathrm{Z}^1(\tilde{\sigma}; \mathcal{V})$, where $w^l(t_1, t_2, g) = w(t_1, g)$, $w^r(t_1, t_2, g) = w(t_2, g)$, $t_1, t_2 \in X$, $g \in \Gamma$.

The above theorem holds in fact true under more general assumptions, in a non-commutative setting. To state the result, we need some notations. Thus, let $\sigma : \Gamma \to \mathrm{Aut}(P, \tau_P)$ be an action of a discrete group Γ on a finite von Neumann algebra (P, τ_P). We assume there exists an extension of σ to an action $\tilde{\sigma}$ of Γ to a larger finite von Neumann algebra $(\tilde{P}, \tilde{\tau})$, with the property that there exists $\alpha_1 \in \mathrm{Aut}(\tilde{P}, \tilde{\tau})$ commuting with the action $\tilde{\sigma}$ and satisfying the properties:

$$\overline{\mathrm{sp}}^w P\alpha_1(P) = \tilde{P}, \tilde{\tau}(x\alpha_1(y)) = \tau(x)\tau(y), \forall x, y \in P. \tag{3.2.0.a}$$

Let (N, τ_N) be another finite von Neumann algebra and ρ an action of Γ on (N, τ). Denote by $\tilde{\sigma}'$ the "diagonal product" action of Γ on $(\tilde{P}\overline{\otimes}N, \tau)$, given by $\tilde{\sigma}'_g = \tilde{\sigma}_g \otimes \rho_g$, $g \in \Gamma$. Notice that the restriction to $P\overline{\otimes}N$ of $\tilde{\sigma}'$ gives the action $\sigma'_g = \sigma_g \otimes \rho_g$, $g \in \Gamma$. Also, denote $M = P\overline{\otimes}N \rtimes_{\sigma'} \Gamma$, $\tilde{M} = \tilde{P}\overline{\otimes}N \rtimes_{\tilde{\sigma}'} \Gamma$, with the corresponding canonical unitaries denoted $\{u_g\}_g \subset M$ respectively $\{\tilde{u}_g\}_g \subset \tilde{M}$. Whenever identifying M with a subalgebra of $\tilde{M}$ in the natural way, we identify $u_g = \tilde{u}_g$.

Note that $[\alpha_1, \tilde{\sigma}] = 0$ implies $[\alpha_1, \tilde{\sigma}'] = 0$ (as usual, we still denote by α_1 the amplifications $\alpha_1 \otimes id_N$). Thus, α_1 extends to an automorphism α_1 of $\tilde{M}$, by $\alpha_1((x \otimes y)\tilde{u}_g) = (\alpha_1(x) \otimes y)\tilde{u}_g$, $\forall x \in \tilde{P}, y \in N$, $g \in \Gamma$.

The inclusion $M \subset \tilde{M}$ and the automorphism α_1 of $\tilde{M}$ implement a $M-M$ Hilbert bimodule structure on $L^2(\tilde{M})$, by

$$z_1 \cdot \xi \cdot z_2 = z_1 \xi \alpha_1(z_2), \forall z_1, z_2 \in M, \xi \in L^2(\tilde{M}), \tag{3.2.0.b}$$

which restricted to $P\overline{\otimes}N$ implements a $P\overline{\otimes}N$ bimodule structure on $L^2(\tilde{P}\overline{\otimes}N)$. With these notations at hand, we have:

3.2. Proposition. *Let $w : \Gamma \to \mathcal{U}(P\overline{\otimes}N)$ be a 1-cocycle for σ', which we also view as a cocycle for $\tilde{\sigma}'$. Assume the action σ of Γ on P is weakly mixing. If there exists $b \neq 0$ in $\tilde{P}\overline{\otimes}N$ satisfying $bw_g = \alpha_1(w_g)\tilde{\sigma}'_g(b)$, $\forall g \in \Gamma$, with the left support of b under some $p_0 \in \mathcal{P}(P\overline{\otimes}N)$, then there exist a non-zero partial isometry $v_0 \in P\overline{\otimes}N$, with $v_0v_0^* \leq p_0$, $p = v_0^*v_0 \in N$, and a local cocycle $w'_g \in \mathcal{U}(pN\rho_g(p))$, $g \in \Gamma$, for ρ such that $w_g\sigma'_g(v_0) = v_0 w'_g, \forall g \in \Gamma$.*

Moreover, $w \sim \alpha_1(w)$ as cocycles for $\tilde{\sigma}'$ if and only if the cocycle w for σ' is equivalent to a cocycle $w' : \Gamma \to \mathcal{U}(N)$ for ρ.

To prove this result we need some further considerations. With the notations introduced above, note that if $g \mapsto v_g \in \mathcal{U}(M)$ is a unitary representation of a group Γ then $\xi \mapsto v_g \cdot \xi \cdot v_g^*$, $\xi \in L^2(\tilde{M})$, $g \in \Gamma$, gives a representation of Γ on the Hilbert space $L^2(\tilde{M})$. Hence, if $\{w_g\}_g \subset \mathcal{U}(P\overline{\otimes}N)$ is a 1-cocycle for σ' and we apply this observation to $v_g = w_g u_g = w_g \tilde{u}_g$ then $\xi \mapsto w_g \mathrm{Ad}(\tilde{u}_g)(\xi)\alpha_1(w_g^*)$, $\xi \in L^2(\tilde{M})$, gives a representation $\tilde{\sigma}'_w$ of Γ on $L^2(\tilde{M})$. This representation clearly leaves $L^2(\tilde{P}\overline{\otimes}N)$ invariant, acting on it by

$$\tilde{\sigma}'_w(g)(\eta) = w_g\tilde{\sigma}'_g(\eta)\alpha_1(w_g^*), \eta \in L^2(\tilde{P}\overline{\otimes}N). \tag{3.2.1}$$

We will now identify the representation $\tilde{\sigma}'_w$ in a different way. Namely, we let $N = 1 \otimes N \subset P\overline{\otimes}N \overset{e_0}{\subset} \langle P\overline{\otimes}N, e_0\rangle$ be the Jones basic construction for the inclusion $N \subset P\overline{\otimes}N$, where e_0 denotes the Jones projection implementing the trace preserving conditional expectation E_N of $P\overline{\otimes}N$ onto N. We then denote by Tr the canonical trace on $\langle P\overline{\otimes}N, e_0\rangle$ and let ${\sigma'}^N$ be the action of Γ on $(\langle P\overline{\otimes}N, e_0\rangle, Tr)$ given by ${\sigma'}^N(g)(z_1e_0z_2) = \sigma'_g(z_1)e_0\sigma'_g(z_2)$ (see 1.4). We also denote

$${\sigma'}^N_w(g)(z_1e_0z_2) = w_g\sigma'_g(z_1)e_0\sigma'_g(z_2)w_g^*, \forall z_1, z_2 \in P\overline{\otimes}N, g \in \Gamma \tag{3.2.2}$$

3.3. Lemma. *Each ${\sigma'}^N_w(g), g \in \Gamma$ defines a Tr-preserving automorphism of the semifinite von Neumann algebra $\langle P\overline{\otimes}N, e_0\rangle$. The map $g \mapsto {\sigma'}^N_w(g)$ gives an action of Γ on $(\langle P\overline{\otimes}N, e_0\rangle, Tr)$, thus also a representation of Γ on the Hilbert space $L^2(\langle P\overline{\otimes}N, e_0\rangle, Tr)$.* ∎

Moreover, the map $x_1\alpha_1(x_2)\otimes y \mapsto (x_1\otimes 1)(ye_0)(x_2\otimes 1)$ *extends to an isomorphism* ϑ_0 *from the Hilbert space* $L^2(\tilde{P\overline{\otimes}N})$ *onto the Hilbert space* $L^2(\langle P\overline{\otimes}N, e_0\rangle, Tr)$ *which intertwines the representations* $\tilde\sigma'_w$, ${\sigma'}_w^N$. *Also,* ϑ_0 *intertwines the left* $P\overline{\otimes}N$*-module structures on these Hilbert spaces.*

While this lemma can be easily given a proof by direct computation, we prefer to derive it as a consequence of a more general result, which has the advantage of giving a clear conceptual explanation of the equivalence of representations in 3.3. More precisely, we will identify ϑ_0 as the restriction (to a subspace) of an isomorphism of Hilbert M-bimodules.

To do this, note that the von Neumann algebra generated in M by $N = 1\otimes N$ and $\{u_g\}_g$ is equal to $N\rtimes\Gamma = N\rtimes_\rho\Gamma$. Let $N\rtimes\Gamma\subset M \overset{e}{\subset} \langle M, e\rangle$ be the basic construction corresponding to the inclusion $N\rtimes\Gamma\subset M$, where e denotes the Jones projection implementing the trace preserving conditional expectation of M onto $N\rtimes\Gamma$. Endow $L^2(\langle M,e\rangle, Tr)$ with the Hilbert M-bimodule structure implemented by the inclusion $M\subset\langle M,e\rangle$. Then we have:

3.4. Lemma. *The map* ϑ *which takes* $(x_1\alpha_1(x_2)\otimes y)\tilde u_g$ *onto* $x_1(yu_g)e\sigma_g^{-1}(x_2)$, $x_1, x_2\in P, y\in N, g\in\Gamma$, *extends to a well defined isomorphism between the Hilbert* M*-bimodules* ${}_ML^2(\tilde M)_M$ *and* ${}_ML^2(\langle M,e\rangle, Tr)_M$.

Proof. Note first that ϑ is consistent with the M-bimodule structures on the two Hilbert spaces. Thus, in order to prove the statement it is sufficient to show that ϑ preserves the scalar product. Let $x_1, x_2, x_1', x_2'\in P$, $y, y'\in N$, $g, g'\in\Gamma$. By the definition of the scalar product in $\tilde M$ and (3.2.0.a), we have:

$$\langle (x_1'\alpha_1(x_2')\otimes y')\tilde u_{g'}, (x_1\alpha_1(x_2)\otimes y)\tilde u_g\rangle = \tau(x_1^*x_1')\tau(x_2^*x_2')\tau(y^*y')\delta_{g',g}$$

On the other hand, by the definition of the scalar product in $L^2(\langle M,e\rangle, Tr)$ we have

$$\langle\vartheta((x_1'\alpha_1(x_2')\otimes y')\tilde u_{g'}), \vartheta((x_1\alpha_1(x_2)\otimes y)\tilde u_g)\rangle$$

$$= \langle x_1'(y'u_{g'})e\sigma_{g'}^{-1}(x_2'), x_1(yu_g)e\sigma_g^{-1}(x_2)\rangle$$

$$= \langle x_1'y'e(x_2'u_{g'}), x_1ye(x_2u_g)\rangle = Tr(E_{N\rtimes\Gamma}(y^*x_1^*x_1'y')eE_{N\rtimes\Gamma}(x_2'u_{g'}u_g^*x_2^*))$$

$$= Tr(y^*E_{N\rtimes\Gamma}(x_1^*x_1')y'eE_{N\rtimes\Gamma}(x_2'\sigma_{g'g^{-1}}(x_2^*))u_{g'g^{-1}})$$

$$= \tau(x_1^*x_1')\tau(x_2'\sigma_{g'g^{-1}}(x_2^*))Tr(y^*y'eu_{g'g^{-1}})$$

$$= \tau(x_1^*x_1')\tau(y^*y')\tau(x_2'\sigma_{g'g^{-1}}(x_2^*))\delta_{g',g}.$$

This shows that ϑ preserves the scalar product, thus finishing the proof. $\square$

Proof of 3.3. We use the notations of 3.4. Since the Jones projection e implements the trace preserving expectation of M onto $N \rtimes_\rho \Gamma$, it also implements the trace preserving expectation of $P\overline{\otimes}N$ onto $N = 1\otimes N$. Moreover, since $\text{sp}P(N \rtimes \Gamma)$ is dense in $L^2(M)$, it follows that the weakly closed * algebra generated by $P\overline{\otimes}N$ and e in $\langle M, e\rangle$ has the same unit as $\langle M, e\rangle$ and is isomorphic to the von Neumann algebra $\langle P\overline{\otimes}N, e_0\rangle$ of the basic construction for $N \subset P\overline{\otimes}N$, with the canonical trace Tr on $\langle P\overline{\otimes}N, e_0\rangle$ corresponding to the restriction to $\langle P\overline{\otimes}N, e\rangle \stackrel{\text{def}}{=} \overline{\text{sp}}^w(P\overline{\otimes}N)e(P\overline{\otimes}N)$ of the canonical trace of $\langle M, e\rangle$. In other words we have a non-degenerate commuting square of inclusions ([P6]):

$$\begin{array}{ccccc} N \rtimes \Gamma & \subset & M & \overset{e}{\subset} & \langle M, e\rangle \\ \cup & & \cup & & \cup \\ N & \subset & P\overline{\otimes}N & \overset{e}{\subset} & \langle P\overline{\otimes}N, e\rangle \end{array}$$

Thus, we can view $L^2(\langle P\overline{\otimes}N, e\rangle, Tr)$ as a Hilbert subspace of $L^2(\langle M, e\rangle, Tr)$. By the definitions, ϑ_0 clearly coincides with the restriction of ϑ to $\tilde{P\overline{\otimes}}N$. The statement follows then by noticing that, by Lemma 3.4, for $g \in \Gamma$ and $\xi \in \tilde{P\overline{\otimes}}N$ we have

$$\vartheta_0(\tilde{\sigma}'_w(g)(\xi)) = \vartheta_0(w_g\tilde{u}_g\xi\tilde{u}_g^*\alpha_1(w_g)^*)$$

$$= \vartheta(w_g\tilde{u}_g\xi\alpha_1(\tilde{u}_g^*w_g^*)) = w_gu_g\vartheta(\xi)u_g^*w_g^*$$

$$= w_gu_g\vartheta_0(\xi)u_g^*w_g^* = {\sigma'}_w^N(g)(\vartheta_0(\xi)).$$

$\square$

Proof of 3.2. By Lemma 3.3, $\xi = \vartheta_0(b) \in L^2(\langle P\overline{\otimes}N, e_0\rangle, Tr)$ is a nonzero element which is fixed by ${\sigma'}_w^N(g), \forall g \in \Gamma$. Moreover, since it preserves the trace Tr, the action ${\sigma'}_w^N$ of Γ on the semifinite von Neumann algebra $(\langle P\overline{\otimes}N, e_0\rangle, Tr)$ induces actions of Γ on all L^p-spaces associated with $(\langle P\overline{\otimes}N, e_0\rangle, Tr)$, being multiplicative whenever a product of elements in such spaces is well defined. This implies that if $\eta = \eta^*$ is say in $L^1(\langle P\overline{\otimes}N, e_0\rangle, Tr)$ and is fixed by ${\sigma'}_w^N$ then all its spectral projections are fixed by ${\sigma'}_w^N$.

From these remarks it follows that $0 \neq \xi\xi^* \in L^1(\langle P\overline{\otimes}N, e_0\rangle, Tr)$ is fixed by ${\sigma'}_w^N$, thus any spectral projection f of $\xi\xi^*$ corresponding to an interval of the form $[c, \infty)$ is fixed by ${\sigma'}_w^N$. Moreover, $Tr(f) < \infty$ whenever $c > 0$, with $f \neq 0$ if $c > 0$ is sufficiently small. Also, since ϑ_0 is a left $P\overline{\otimes}N$-module isomorphism, if $p_0 \in P\overline{\otimes}N \subset \tilde{P\overline{\otimes}}N$ is a projection satisfying $p_0b = p_0$, then $p_0\xi = \xi$ and thus $p_0f = f$.

Let $f = \Sigma_i m_ie_0m_i^*$, for some $\{m_i\}_i \subset L^2(P\overline{\otimes}N, \tau)$ satisfying $E_N(m_i^*m_j) = \delta_{ij}p_j \in \mathcal{P}(N), \forall i, j$ (see e.g. [P6]). In other words, $\{m_i\}_i$ is an orthonormal basis of $\mathcal{H} = f(L^2(P\overline{\otimes}N, \tau))$. We next show that by "cutting" f with an appropriate projection in $\mathcal{Z}(N)^\rho$, we may assume $\{m_i\}_i$ is a finite set.

Since $\tau(\Sigma_i m_i m_i^*) = Trf < \infty$, the operator valued weight $\phi(xe_0y) = xy$ satisfies $\phi(f) = \Sigma_i m_i m_i^* \in L^1(P\overline{\otimes}N, \tau)$. Since

$$\Sigma_i m_i e_0 m_i^* = f = {\sigma'_w}^N(f) = \Sigma_i w_g \sigma'_g(m_i) e_0 \sigma'_g(m_i)^* w_g^*,$$

by applying ϕ to the first and last terms we get $\mathrm{Ad}w_g \circ \sigma'_g(\Sigma_i m_i m_i^*) = \Sigma_i m_i m_i^*$, $\forall g$. Thus $\Sigma_i m_i m_i^*$ is fixed by the action $\lambda_g = \mathrm{Ad}w_g \circ \sigma'_g$ of Γ on $(P\overline{\otimes}N, \tau)$. Also, λ leaves invariant the centers of $P\overline{\otimes}N$ and N (the latter because σ' leaves N invariant and $\mathrm{Ad}w_g$ commutes with $\mathcal{Z}(N) \subset \mathcal{Z}(P\overline{\otimes}N) = Z(N)\overline{\otimes}\mathcal{Z}(N)$). Thus, $a = E_{\mathcal{Z}(N)}(\Sigma_i m_i m_i^*)$ is fixed by λ and so are all its spectral projections. Let z be a spectral projection of a such that $0 \neq \|za\| < \infty$.

Since $\mathcal{Z}(N) \subset \mathcal{Z}(\langle P\overline{\otimes}N, e_0\rangle)$, z commutes with $\{m_i\}_i$, e and f. Thus, by replacing f with zf we may assume the orthonormal basis $\{m_i\}_i$ is so that $E_{\mathcal{Z}(N)}(\Sigma_i m_i m_i^*)$ is a bounded operator. But $E_{\mathcal{Z}(N)} = E_{\mathcal{Z}(N)} \circ E_{\mathcal{Z}(P\overline{\otimes}N)}$, the latter being in fact the central trace Ctr on $P\overline{\otimes}N$. Since $Ctr(xy) = Ctr(yx)$, we get $Ctr(\Sigma_i m_i m_i^*) = Ctr(\Sigma_i m_i^* m_i)$ and thus $E_{\mathcal{Z}(N)}(\Sigma_i m_i^* m_i) = E_{\mathcal{Z}(N)}(\Sigma_i m_i m_i^*)$ is bounded. This implies $p_i = E_N(m_i^* m_i) \in \mathcal{P}(N)$ satisfy $\Sigma_i Ctr_N(p_i)$ bounded, where Ctr_N denotes the central trace on N. Thus, by using the "cutting and gluing" procedures in (1.1.4 of [P6]) we can choose the orhonormal basis $m_1, m_2, ...$ such that the central supports $z(p_i)$ of p_i in N satisfy $p_i \geq z(p_{i+1}), \forall i$. But this implies that the cardinality of the set $\{m_2, m_3, ...\}$ is majorized by $\|\Sigma_i Ctr(p_i)\| < \infty$, thus showing that it is a finite set.

Let t be the cardinality of the finite orthonormal basis $\{m_i\}_i$. Denote $N^t = M_{t\times t}(N) = N \otimes M_{t\times t}(\mathbb{C})$ and notice that by tensoring all inclusions $N \subset P\overline{\otimes}N \subset \langle P\overline{\otimes}N, e_0\rangle$ by $M_{t\times t}(\mathbb{C})$ one gets the non-degenerate commuting squares of inclusions

$$\begin{array}{ccccc} N^t & \overset{E_N^t}{\subset} & P\overline{\otimes}N^t & \overset{e_0}{\subset} & \langle P\overline{\otimes}N^t, e_0\rangle \\ \cup & & \cup & & \cup \\ N & \overset{E_N}{\subset} & P\overline{\otimes}N & \overset{e_0}{\subset} & \langle P\overline{\otimes}N, e_0\rangle \end{array}$$

with ${\sigma'}^N$ extending to an action ${\sigma'}^{N^t}$ of Γ on $\langle P\overline{\otimes}N^t, e_0\rangle$, which acts as $\rho \otimes id$ on $N \otimes M_{t\times t}(\mathbb{C}) = N^t$. Similarly, we denote ${\sigma'_w}^{N^t}$ the action $\mathrm{Ad}w_g \circ {\sigma'}^{N^t}(g), g \in \Gamma$.

Let $f_0 = fe_{11}$, where $\{e_{ij}\}_{i,j}$ are matrix units for $M_{t\times t}(\mathbb{C}) \subset 1 \otimes N^t \subset P\overline{\otimes}N^t$. Note that $f_0 \prec e_0$ in $\langle P\overline{\otimes}N^t, e_0\rangle$, so there exists an element $m \in L^2(P\overline{\otimes}N^t)$ such that $f_0 = me_0m^*$. In particular, $q = E_{N^t}(m^*m)$ is a projection in $N^t = 1_P \otimes N^t$

Since $me_0m^* = {\sigma'}_w^{N^t}(g)(me_0m^*) = w_g {\sigma'}_g(m) e_0 \sigma'_g(m^*) w_g^*$, it follows that $me_0N^t = w_g\sigma'_g(m)e_0N^t$, $\forall g \in \Gamma$. Taking the operator valued weight of $\langle P\overline{\otimes}N^t, e_0\rangle$ onto $P\overline{\otimes}N^t$, it follows that $mN^t = w_g\sigma'_g(m)N^t$, $\forall g \in \Gamma$, as well. Thus, for each $g \in \Gamma$ there exists $w'_g \in N^t$ such that $mw'_g = w_g\sigma'_g(m)$ ($w'_g = E_{N^t}(m^*w_g\sigma'_g(m)$ will do), and we have

$$\sigma'_g(m^*m) = \sigma'_g(m^*)w_g^*w_g\sigma'_g(m) = {w'_g}^* m^*mw'_g, \forall g.$$

Since σ'_g commutes with E_{N^t}, by applying E_{N^t} to these equalities we get $\sigma'_g(q) = {w'_g}^* q w'_g$, $\forall g$, showing that $q w'_g \rho_g(q)$ are in the set $\mathcal{U}(qN^t\rho_g(q))$ of partial isometries with left support equal to q and right support equal to $\rho_g(q)$. Thus, by replacing w'_g by $qw'_g\rho(q)$ we may assume $w'_g \in \mathcal{U}(qN^t\rho_g(q))$, $\forall g$.

Since $w_h\sigma'_h(m) = mw'_h$, $w_{gh}\sigma'_{gh}(m) = mw'_{gh}$ and $w_{gh} = w_g\sigma'_g(w_h)$, we then get for all $g, h \in \Gamma$:

$$mw'_{gh} = w_{gh}\sigma'_{gh}(m) = w_{gh}\sigma'_g(\sigma'_h(m))$$

$$= w_g\sigma'_g(w_h)\sigma'_g(\sigma'_h(m)) = w_g\sigma'_g(w_h\sigma'_h(m)) = w_g\sigma'_g(mw'_h)$$

$$= w_g\sigma'_g(m)\rho_g(w'_h) = mw'_g\rho_g(w'_h)$$

showing that $w'_{gh} = w'_g\rho_g(w'_h)$, i.e. w' is a local cocycle for ρ.

Note that if one denotes by π_1 (resp. π_2) the representations of Γ into the unitary group of $M = P\overline{\otimes}N^t \rtimes \Gamma$ (resp. into the unitary group of qMq) given by $\pi_1(h) = w_hu_h$, $h \in \Gamma$ (resp. $\pi_2(g) = w'_gu_g$, $g \in \Gamma$), then the relation $w_g\sigma_g(m) = mw'_g, \forall g$, simply states that π_1, π_2 are intertwined by m. This implies that the partial isometry $v \in P\overline{\otimes}N^t$ in the polar decomposition of m intertwines these representations as well and that v^*v commutes with $\pi_2(\Gamma)$. Thus, v^*v belongs to the subalgebra of $P\overline{\otimes}qN^tq$ fixed by the action $\sigma_g \otimes (\mathrm{Ad}(w'_g) \circ \rho_g), g \in \Gamma$, which since σ_g is weakly mixing follows equal to the fixed point algebra B_0 of the action $\mathrm{Ad}w'_g \circ \rho_g$ of Γ on qN^tq.

On the other hand, since $e_{11}m = m$ it follows that $v = e_{11}v$, implying that the central trace of v^*v in N^t is $\leq 1/t$. Thus, v^*v is equivalent in N^t to a projection of the form $q_0e_{11} \in N^t$ with $q_0 \in N = N \otimes 1 \subset N^t$. Let $u_0 \in \mathcal{U}(N^t)$ be so that $u_0q_0e_{11}u_0^* = v^*v$.

Denote $v_1 = vu_0$, $w'_1(g) = u_0^*w'_g\rho_g(u_0)$. We then have for all $g \in \Gamma$:

$$v_1w'_1(g) = vu_0u_0^*w'_g\rho_g(u_0) = vw'_g\rho_g(u_0)$$

$$= w_g\sigma'_g(v)\rho_g(u_0) = w_g\sigma'_g(vu_0) = w_g\sigma'_g(v_1).$$

Also, $v_1 = e_{11}v_1e_{11}$, $v_1^*v_1 \in e_{11}N^te_{11} = Ne_{11}$ and $w'_1(g) \in \mathcal{U}(v_1^*v_1N^t\rho_g(v_1^*v_1))$. This implies there exists a partial isometry $v_0 \in N$ and a cocycle $w'_0 : \Gamma \to \mathcal{U}(v_0^*v_0N\rho_g(v_0^*v_0))$ for ρ such that $v_1 = v_0e_{11}$ and $w'_1(g) = w'_0(g)e_{11}, \forall g$. But then, by the isomorphism $N \simeq Ne_{11}$, we clearly have $v_0w'_0(g) = w_g\sigma'_g(v_0), \forall g$. Finally, notice that by the definition of m and the fact that $p_0f = p_0$, we have $p_0m = p_0$, implying that $v_0v_0^* \leq p_0$.

This proves the first part of 3.2. To prove the last part, we use a maximality argument. Thus, denote by $\mathscr{W}$ the set of pairs (v, w') with $v \in P\overline{\otimes}N$ partial isometry satisfying $v^*v \in N = 1 \otimes N$ and $w' : \Gamma \to \mathcal{U}(v^*vN\rho_g(v^*v))$ a local cocycle for ρ such that $vw'(g) = w_g\sigma'_g(v)$, $\forall g \in \Gamma$. We endow $\mathscr{W}$ with the order: $(v_0, w'_0) \leq (v_1, w'_1)$ iff $v_0 = v_1v_0^*v_0$, $v^*vw'_1(g) = w'_0(g), \forall g \in \Gamma$.

$(\mathscr{W}, \leq)$ is clearly inductively ordered, so let $(v_0, w_0') \in \mathscr{W}$ be a maximal element. We want to prove that v_0 is a unitary element (automatically implying that w_0' is a a cocycle for ρ). Assume v_0 is not a unitary element and let $v = v_0\alpha_1(v_0)^* \in \tilde{P}\overline{\otimes}N$. Since α_1 acts as the identity on N, it follows that $vv^* = v_0v_0^*$. Also, for $g \in \Gamma$ we have

$$w_g\tilde{\sigma}'_g(v) = w_g\sigma'_g(v_0)\alpha_1(\sigma'_g(v_0))^* = v_0w_0'(g)\alpha_1(\sigma'_g(v_0^*))$$

$$= v_0\alpha_1(\sigma'_g((v_0w_0'(g^{-1}))^*)) = v_0\alpha_1(\sigma'_g(w_{g^{-1}}\sigma'_{g^{-1}}(v_0))^*))$$

$$= v_0\alpha_1(\sigma'_g(\sigma'_{g^{-1}}(v_0^*)\sigma'_g(w_{g^{-1}})^*)) = v_0\alpha_1(v_0^*)\alpha_1(w_g),$$

where for the last equality we have used the fact that the cocycle relation $w_g\sigma'_g(w_{g^{-1}}) = w_e = 1$ implies $\sigma'_g(w_{g^{-1}})^* = w_g$. Thus, since w and $\alpha_1(w)$ are equivalent cocycles for the action $\tilde{\sigma}'$, by 2.9 it follows that there exists a partial isometry $v' \in \tilde{P}\overline{\otimes}N$ such that $w_g\tilde{\sigma}'_g(v') = v'\alpha_1(w_g)$, $\forall g \in \Gamma$, and $v'v'^* = 1 - vv^*$, $v'^*v' = 1 - v^*v$. The first part then implies there exists a non-zero partial isometry $v_1 \in P\overline{\otimes}N$, with left support majorized by $1 - v_0v_0^*$ and right support in N, such that $w_g\sigma'_g(v_1) = v_1w_1'(g)$, $\forall g \in \Gamma$, for some local cocycle $w_1' : \Gamma \to \mathcal{U}(v_1^*v_1N\rho_g(v_1^*v_1))$ for ρ.

Thus, in the finite von Neumann algebra $\tilde{P}\overline{\otimes}N$ we have the equivalence of projections $v_1^*v_1 \sim v_1v_1^* \prec 1 - v_0v_0^* \sim 1 - v_0^*v_0$. Since the first and last projections lye in N and satisfy $v_1^*v_1 \prec 1 - v_0^*v_0$ in $\tilde{P}\overline{\otimes}N$, they satisfy the same relation in N (for instance, because the central trace of $\tilde{P}\overline{\otimes}N$ is the tensor product of the central traces on $\tilde{P}$ and N). Thus, by multiplying v_1 to the right with a partial isometry in N that makes $v_1^*v_1$ equivalent to a projection in $(1 - v_0^*v_0)N(1 - v_0^*v_0)$ and conjugating w_1' appropriately, we may assume $v_1^*v_1 \leq 1 - v_0^*v_0$. But then $(v_0 + v_1, w_0' \oplus w_1') \in \mathscr{W}$ strictly majorizes (v_0, w_0'), contradicting the maximality of (v_0, w_0').

$\square$

The next result shows that untwisting cocycles with values in Polish groups of finite type is a hereditary property. The rather elementary argument is reminiscent of the proof of (5.2 in [P3]).

3.5. Proposition. *Let σ be a weakly mixing action of the discrete group Γ on the standard probability space (X, μ). Let $\mathcal{W} \in \mathscr{U}_{fin}$ and $\mathcal{V} \subset \mathcal{W}$ a closed subgroup. If a $\mathcal{V}$-valued cocycle for σ can be untwisted as a $\mathcal{W}$-valued cocycle then it can be untwisted as a $\mathcal{V}$-valued cocycle. More generally, let ρ be another action of Γ on a standard probability space (Y, ν). If a $\mathcal{V}$-valued cocycle w for the diagonal product action $\sigma \times \rho$ is equivalent to a $\mathcal{W}$-valued cocycle for ρ then it is equivalent to $\mathcal{V}$-valued cocycle for ρ.*

Proof. Embed $\mathcal{W}$ as a closed subgroup of the unitary group of a separable finite von Neumann algebra (N, τ). Let $w : \Gamma \to \mathcal{V}^{X \times Y}$ be a $\mathcal{V}$-valued 1-cocycle for σ and assume

there exist a cocycle $w' : \Gamma \to \mathcal{W}^Y$ for ρ and a measurable map $v : X \times Y \to \mathcal{W}$ such that for each $g \in \Gamma$ we have

$$w(t,s,g)v(g^{-1}t, g^{-1}s) = v(t,s)w'(s,g), \forall (t,s) \in X \times Y (a.e.) \tag{3.5.1}$$

By the approximation Lemma 2.2 applied to the measurable map $X \ni t \mapsto v(t,\cdot) \in \mathcal{W}^Y$, there exists a decreasing sequence of subsets of positive measure $X \supset X_1 \supset ...$ and an element $u_0 \in \mathcal{W}^Y$ such that $\|v(t) - u_0\|_2 \leq 2^{-n}$, $\forall t \in X_n$, $\forall n \geq 1$, where $\|\cdot\|_2$ corresponds here to the Hilbert norm on $L^\infty Y \overline{\otimes} N$ given by $\tau_\nu \otimes \tau$.

This shows that by replacing v, regarded as a function in $t \in X$, by $v(t)u_0^*, t \in X$, and $w'(\cdot, g)$ by $u_0 w'(\cdot, g)\rho(u_0^*)$, $g \in \Gamma$, we may assume $v \in \mathcal{W}^{X \times Y}$ and $w' \in \mathrm{Z}^1(\rho; \mathcal{W})$ satisfy both (3.5.1) and

$$\|v(t) - 1\|_2 \leq 2^{-n}, \forall t \in X_n, \forall n \geq 1, \tag{3.5.2}$$

for some decreasing sequence of subsets of positive measure X_n. We'll show that, together with the weak-mixing assumption, this entails $v(t) \in \mathcal{V}^Y$, $\forall t \in X$ (a.e.) and $w'(\Gamma) \subset \mathcal{V}^Y$, where $v(t) = v(t, \cdot)$ and $w'(g) = w'(\cdot, g)$.

Thus, let $h_n \to \infty$ in Γ be so that $\lim_n \mu(h_n Z \cap Z') = \mu(Z)\mu(Z')$, $\forall Z, Z' \subset X$ measurable. Note that this implies that for any $g, g' \in \Gamma$ the sequences $\{g h_n g'\}_n, \{g h_n^{-1} g'\}_n$ satisfy the same condition.

Fix $h \in \Gamma$. For each $m \geq 1$ let n_m be so that $X'_m = X_m \cap h_{n_m} X_m$ and $X''_m = X_m \cap (h_{n_m} h) X_m$ have positive measure. Then for t' in X'_m we have $\|v(t') - 1\|_2 \leq 2^{-m}$, $\|v(h_{n_m}^{-1} t') - 1\|_2 \leq 2^{-m}$ while for $t'' \in X''_m$ we have $\|v(t'') - 1\|_2 \leq 2^{-m}$ and $\|v((h_{n_m} h)^{-1} t'') - 1\|_2 \leq 2^{-m}$. By applying (3.5.1) first for $g = h_{n_m}$ and then for $g = h_{n_m} h$, if we denote $w(t,g) = w(t, \cdot, g) \in \mathcal{V}^Y$, this implies

$$\|w(t', h_{n_m}) - w'(h_{n_m})\|_2 \leq 2^{-m+1}, \forall t' \in X'_m$$

and respectively

$$\|w(t'', h_{n_m} h) - w'(h_{n_m} h)\|_2 \leq 2^{-m+1}, \forall t'' \in X''_m.$$

But this entails

$$\begin{aligned} &\|\rho_{h_{n_m}}(w'(h)) - w(t', h_{n_m})^* w(t'', h_{n_m} h)\|_2 \\ &= \|w(t', h_{n_m})\rho_{h_{n_m}}(w'(h)) - w(t'', h_{n_m} h)\|_2 \\ &\leq \|w(t', h_{n_m})\rho_{h_{n_m}}(w'(h)) - w'(h_{n_m})\rho_{h_{n_m}}(w'(h))\|_2 \\ &+ \|w'(h_{n_m} h) - w(t'', h_{n_m} h)\|_2 \leq 2^{-m+2} \end{aligned}$$

showing that $w'(h)$ can be approximated arbitrarily well with elements in $\mathcal{V}$. Thus $w'(h) \in \mathcal{V}$.

We still need to prove that $v(t) \in \mathcal{V}^Y$, $\forall t \in X(a.e.)$. By using the approximation Lemma 2.2, it is clearly sufficient to prove that for any set $Z' \subset X$ of positive measure and any $\varepsilon > 0$ there exists a subset of positive measure $Z_0 \subset Z'$ such that $\mathrm{d}(v(t), \mathcal{V}^Y) < \varepsilon$, $\forall t \in Z_0$. Let $h_n \in \Gamma$ be so that $\lim_n \mu(h_n Z \cap Z') = \mu(Z)\mu(Z')$, $\forall Z, Z' \subset X$ measurable, as before. By (3.5.2) we can take m sufficiently large such that $\|v(t)-1\|_2 \leq \varepsilon$, $\forall t \in X_m$. Let $n = n(m)$ be so that $\mu(h_n X_m \cap Z') \neq 0$ and denote $Z_0 = h_n X_m \cap Z'$. If t_0 belongs to Z_0 then $t = h_n^{-1} t_0 \in X_n$ so if we apply (3.5.1) to $g = h_n^{-1}$ we get the following identity in $\mathcal{W}^Y$:

$$v(t_0, h_n \cdot) = w(t, \cdot, h_n^{-1})^{-1} v(t, \cdot) w'(\cdot, h_n^{-1})$$

Since both $w(t, \cdot, h_n^{-1})$ and $w'(\cdot, h_n^{-1})$ lie in $\mathcal{V}^Y$ and $v(t, \cdot)$ is ε-close to an element in $\mathcal{V}^Y$, it follows that $v(t_0, h_n \cdot)$ is ε-close to an element in $\mathcal{V}^Y$, with $t_0 \in Z_0$ arbitrary. Thus, $v(t_0, \cdot) \in \mathcal{V}^Y$, $\forall t_0 \in Z_0$, as well $\square$

Proof of Theorem 3.1. Since $\mathcal{V} \in \mathscr{U}_{fin}$, the group $\mathcal{V}$ is isomorphic to a closed subgroup of $\mathcal{U}(N)$, for some separable II_1 factor N. By Proposition 3.2 it follows that, as cocycles for $\sigma \times \rho$, w is equivalent to a $\mathcal{U}(N)$-valued cocycle w' for ρ. But by Propositon 3.5, w is then equivalent in $\mathrm{Z}^1(\sigma \times \rho; \mathcal{V})$ to a cocycle lying in $\mathrm{Z}^1(\rho; \mathcal{V})$. $\square$

3.6. Proposition. *Let σ, ρ be actions of the discrete group Γ on finite von Neumann algebras $(P, \tau), (N, \tau)$ and w a cocycle for $\sigma \otimes \rho$. Let $H \subset \Gamma$ be an infinite subgroup and assume $w_h \in \mathcal{U}(N), h \in H$. Let H' denote set set of all $g \in \Gamma$ for which $w_g \in N$. Then we have:*

1°. H' is a subgroup of Γ.

2°. If $g \in \Gamma$ is so that $gHg^{-1} \cap H$ is infinite and σ is weak mixing on $gHg^{-1} \cap H$ then $g \in H'$.

3°. If H is normal in Γ and σ is weak mixing on H then $H' = \Gamma$.

4°. If σ is weak mixing on Γ then two cocycles $w_1, w_2 : \Gamma \to \mathcal{U}(N)$ for ρ are equivalent as cocycles for the diagonal product action $\sigma \otimes \rho$ if and only if they are equivalent as cocycles for ρ.

Proof. Part 1° is trivial and 3° follows from 2°.

To prove 2° we apply Lemma 2.10 to the action λ of $H_0 = H \cap gHg^{-1}$ on $P \overline{\otimes} N$ given by $\lambda_{h_0} = \mathrm{Ad}(w_{h_0}) \circ (\sigma_{h_0} \otimes \rho_{h_0}), h_0 \in H_0$. To this end, let us note that if h_0 is an arbitrary element in H_0 and we denote $h = g^{-1} h_0 g$ then $h \in H$ and $h_0 = ghg^{-1}$. By using the cocycle relations for w we then get:

$$\lambda_{h_0}(w_g) = w_{h_0} (\sigma_{h_0} \otimes \rho_{h_0})(w_g) w_{h_0}^*$$

$$= w_{h_0 g} w_{h_0}^* = w_{gh} w_{h_0}^* = w_g (\sigma_g \otimes \rho_g)(w_h) w_{h_0}^* = w_g \rho_g(w_h) w_{h_0}^*.$$

Thus, $\lambda_{h_0}(w_g N) = w_g N$, $\forall h_0 \in H_0$. Since the action λ of H_0 is weak mixing on $P = P \otimes 1$ and leaves $N = 1 \otimes N$ globally invariant, by 2.10 it follows that $w_g \in 1 \otimes N$, i.e. $g \in H'$.

4°. Assume there exists $v \in \mathcal{U}(P\overline{\otimes}N)$ such that $v^* w_1(g)(\sigma_g \otimes \rho_g)(v) = w_2(g)$ for all $g \in \Gamma$. Thus, $\mathrm{Ad}w_1(g)((\sigma_g \otimes \rho_g)(v)) = v w_2(g) w_1(g)^*$, where we have identified $w_i(g)$ with $1 \otimes w_i(g)$. This shows that $\lambda_g = \mathrm{Ad}(1 \otimes w_1(g)) \circ (\sigma_g \otimes \rho_g)$ leaves invariant the space $v(1 \otimes N)$. By 2.10 this implies $v \in 1 \otimes N$. $\square$

4. Perturbation of cocycles and the use of property (T)

In this section we prove that if a Γ-action σ has good deformation properties and the group satisfies a weak form of the property (T), then any cocycle for σ checks the untwisting criterion 3.1. The proof uses the same "deformation/rigidity" argument as on (page 304 of [P1]) and (page 31 of [P2]).

4.1. Definition. An inclusion of discrete groups $H \subset \Gamma$ has the *relative property* (T) of Kazhdan-Margulis (or is *rigid*) if the following holds true (cf [Ma]; see also [dHV]):

(4.1.1). $\exists F_0 \subset \Gamma$ finite and $\varepsilon_0 > 0$ such that if π is a unitary representation of Γ on a Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi_g(\xi) - \xi\| \leq \varepsilon_0$, $\forall g \in F_0$, then $\exists 0 \neq \xi_0 \in \mathcal{H}$ with $\pi_h(\xi_0) = \xi_0, \forall h \in H$.

Note that in case $H = \Gamma$, (4.1.1) amounts to the usual property (T) of Kazhdan for Γ ([K]). It is easy to see that if $H = \Gamma$, more generally if H is normal in Γ, then the fixed vector ξ_0 in (4.1.1) can be taken close to ξ. This fact was shown in ([Jo]) to hold true for arbitrary (not necessarily normal) inclusions $H \subset \Gamma$ with the relative property (T). In turn, the characterization of the relative property (T) for $H \subset \Gamma$ in ([Jo]) is easily seen to be equivalent to the following property, more suitable for us here:

(4.1.2). $\forall \varepsilon > 0$ there exist a finite subset $F(\varepsilon) \subset \Gamma$ and $\delta(\varepsilon) > 0$ such that if $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ on the Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g)\xi - \xi\| < \delta(\varepsilon)$, $\forall g \in F(\varepsilon)$, then $\|\pi_h(\xi) - \xi\| < \varepsilon$, $\forall h \in H$.

The relative property (T) will only be used through the following:

4.2. Lemma. *Let σ be an action of a discrete group Γ on a finite von Neumann algebra (Q, τ). Assume $H \subset \Gamma$ is a subgroup with the relative property* (T)*. If w is a cocycle for σ, then for any $\varepsilon > 0$ there exists a neighborhood Ω of w in the space $\mathrm{Z}^1(\sigma)$ of cocycles for σ such that $\forall w' \in \Omega$ $\exists v \in Q$ partial isometry satisfying $w'_h \sigma_h(v) = v w_h, \forall h \in H$ and $\|v - 1\|_2 \leq \varepsilon$. Moreover, if the action $\mathrm{Ad}(w_h) \circ \sigma_h, h \in H$, of H on (Q, τ) is ergodic, then the restriction to H of any cocycle in Ω is cohomologous to $w_{|H}$ (as cocycles for $\sigma_{|H}$).*

Proof. With the notations in 4.1, let $F = F(\varepsilon)$ and $\delta = \delta(\varepsilon^2/4)$. Let $w' \in \mathrm{Z}^1(\sigma)$ be so that $\|w_g - w'_g\|_2 \leq \delta$, $\forall g \in F$. Let π be the representation of Γ on L^2Q defined by $\pi(g)(\eta) = w'_g \sigma_g(\eta) w^*_g$, $\eta \in L^2Q$, $g \in \Gamma$.

Then we have

$$\|\pi_g(1)-1\|_2 = \|w'_g\sigma_g(1)w_g^* - 1\|_2 = \|w_g - w'_g\|_2 \le \delta, \forall g \in F.$$

Thus, $\|\pi_h(1)-1\|_2 \le \varepsilon^2/4$. Letting $\xi_0 \in L^2Q$ be the element of minimal norm $\|\cdot\|_2$ in $K = \overline{\mathrm{co}}^w\{\pi_h(1) \mid h \in H\}$, it follows that $\|\xi_0 - 1\|_2 \le \varepsilon^2/4$, $\|\xi_0\| \le 1$ (thus $\xi_0 \in Q$) and $w'_h\sigma_h(\xi_0)w_h^* = \xi_0$, $\forall h \in H$. But then $w'_h\sigma_h(\xi_0) = \xi_0 w_h$, $\forall h \in H$, so if $v \in Q$ denotes the partial isometry in the polar decomposition of ξ_0 then $w'_h\sigma_h(v) = vw_h$, $\forall h \in H$, and by ([C2]) we have $\|v-1\|_2 \le \varepsilon$.

Thus, if we let $\Omega = \{w' \in \mathrm{Z}^1(\sigma) \mid \|w'_g - w_g\|_2 \le \delta, \forall g \in F\}$, then Ω satisfies the desired conditions.

Now, since the relation $w'_h\sigma_h(v) = vw_h$, $\forall g \in H$, implies that v^*v is in the fixed point algebra of the action $\mathrm{Ad}w_h \circ \sigma_h, h \in H$, of H on Q, if the cocycle w is ergodic then $v^*v = 1$ showing that $w' \sim w$ as cocycles for $\sigma_{|H}$. □

4.3. Definition. Let σ be an action of a discrete group Γ on a standard probability space (X,μ). Denote by $(A,\tau) = (L^\infty(X,\mu), \int \cdot \mathrm{d}\mu)$ the associated abelian von Neumann algebra and by $\tilde\sigma : \Gamma \to \mathrm{Aut}(A\overline{\otimes}A, \tau\otimes\tau)$ the diagonal product action given by $\tilde\sigma_g = \sigma_g \otimes \sigma_g$, $g \in \Gamma$. The action σ is *malleable* if the flip automorphism $\alpha' \in \mathrm{Aut}(A\overline{\otimes}A, \tau\otimes\tau)$, defined by $\alpha'(a_1\otimes a_2) = a_2 \otimes a_1$, $a_1, a_2 \in A$, is in the (path) connected component of the identity in the commutant of $\tilde\sigma(\Gamma)$ in $\mathrm{Aut}(A\overline{\otimes}A, \tau\otimes\tau)$ (cf [P4]; note that this terminology is used in [P1,2,3] for a slightly stronger condition).

The action σ is *s-malleable* if there exist a continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}(A\overline{\otimes}A, \tau\otimes\tau)$ and a period 2 automorphism $\beta \in \mathrm{Aut}(A\overline{\otimes}A, \tau\otimes\tau)$ satisfying

$$[\alpha, \tilde\sigma] = 0, \alpha_1(A\otimes 1) = 1 \otimes A, \tag{4.3.1}$$

$$[\beta, \tilde\sigma] = 0, \beta(a\otimes 1) = a\otimes 1, \forall a \in A, \beta\alpha_t = \alpha_{-t}\beta, \forall t \in \mathbb{R}, \tag{4.3.2}$$

More generally, an action σ of Γ on a finite von Neumann algebra (P,τ) is *s-malleable* if there exist a continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}(P\overline{\otimes}P, \tau\otimes\tau)$ and a period 2 automorphism $\beta \in \mathrm{Aut}(P\overline{\otimes}P, \tau\otimes\tau)$ such that if we denote by $\tilde\sigma$ the diagonal product action $\tilde\sigma_g = \sigma_g \otimes \sigma_g$ of Γ on $(P\overline{\otimes}P, \tau\otimes\tau)$ then we have:

(4.3.1′). α commutes with $\tilde\sigma$ and satisfies:

$$\tau(x\alpha_1(x)) = \tau(x)\tau(y), \forall x, y \in P; \overline{\mathrm{sp}}^w P\alpha_1(P) = P\overline{\otimes}P.$$

(4.3.2′). β commutes with $\tilde\sigma$ and satisfies

$$\beta(x\otimes 1) = x\otimes 1, \forall x \in P, \beta\alpha_t = \alpha_{-t}\beta, \forall t \in \mathbb{R}.$$

4.4. Example. Let (X_0, μ_0) be a standard probability space. Let Γ be a countable discrete group and K a countable set on which Γ acts. Let $(X, \mu) = \Pi_k (X_0, \mu_0)_k$ be the standard probability space obtained as the product of identical copies $(X_0, \mu_0)_k$ of (X_0, μ_0), $k \in K$. Let $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$ be defined by $\sigma(g)((x_k)_k) = (x'_k)_k$, where $x'_k = x_{g^{-1}k}$. We call σ the (X_0, μ_0)-*Bernoulli* $(\Gamma \curvearrowright K)$-*action.* We generically refer to such actions as *generalized Bernoulli actions.* In case $K = \Gamma$ and $\Gamma \curvearrowright \Gamma$ is the left multiplication, we simply call σ the (X_0, μ_0)-*Bernoulli* Γ-*action.*

Note that if we denote $(A_0, \tau_0) = (L^\infty X_0, \int \cdot \mathrm{d}\mu_0)$, then the algebra $(L^\infty X, \int \cdot \mathrm{d}\mu)$ coincides with $\overline{\otimes}_k (A_0, \tau_0)$, with the action implemented by σ on elements of the form $\otimes_k a_k \in \overline{\otimes}_k (A_0, \tau_0)$ being given by $\sigma_g(\otimes_k a_k) = \otimes_k a'_k$, $a'_k = a_{g^{-1}k}, k \in K$, $g \in \Gamma$.

More generally, if (N_0, τ_0) is a finite von Neumann algebra and we denote $(N, \tau) = \overline{\otimes}_k (N_0, \tau_0)_k$, then for each $g \in \Gamma$, $\otimes_k a_k \in N$ we let $\sigma_g(\otimes_k a_k) = \otimes_k a'_k$, where $a'_k = a_{g^{-1}k}, k \in K$. Then σ is clearly an action of Γ on (N, τ) which we call the (N_0, τ_0)-*Bernoulli* $(\Gamma \curvearrowright K)$-*action.*

It was shown in ([P1], [P2]) that if N_0 is either abelian diffuse or a tensor product of 2 by 2 matrix algebras then any (N_0, τ_0)-Bernoulli $\Gamma \curvearrowright K$ action is s-malleable. We reprove these results below, for the sake of completeness.

4.5. Lemma. *Let Γ be a countable discrete group acting on a countable set K. Let (N_0, τ_0) be a finite von Neumann algebra and σ the (N_0, τ_0)-Bernoulli $\Gamma \curvearrowright K$ action on $(N, \tau) = \overline{\otimes}_{k \in K} (N_0, \tau_0)_k$.*

1°. *If either (N_0, τ_0) is diffuse (i.e. it has no non-zero minimal projections) and for all $g \neq e$ there exists $k \in K$ such that $gk \neq k$, or if (N_0, τ_0) is arbitrary and for all $g \neq e$ the set $\{k \in K \mid gk \neq k\}$ is infinite, then σ is a free action.*

2°. *σ is weakly mixing iff $\forall K_0 \subset K$ $\exists g \in \Gamma$ such that $gK_0 \cap K_0 = \emptyset$ and iff any orbit of $\Gamma \curvearrowright K$ is infinite. Moreover, if any of these conditions is satisfied then σ is weakly mixing relative to any subalgebra of the form $N^0 = \overline{\otimes}_k (N_0^0)_k \subset N$, with $N_0^0 \subset N_0$.*

3°. *σ is (strongly) mixing iff $\forall K_0 \subset K$ finite $\exists F \subset \Gamma$ finite such that $gK_0 \cap K_0 = \emptyset$, $\forall g \in \Gamma \setminus F$, and iff the stabilizer $\{h \in \Gamma \mid hk = k\}$ of any $k \in K$ is finite.*

4°. *If N_0 is either abelian diffuse or a finite factor of the form $(N_0, \tau_0) = \overline{\otimes}_{l \in L} (M_{2 \times 2}(\mathbb{C}), tr)_l$, for some set of indices L, then σ is s-malleable.*

Proof. 1°, 3° and the first part of 2° are well known (and easy exercises!). To prove the last part of 2°, let $\{\eta_n \mid n \geq 0\} \subset L^2(N_0, \tau_0)$ be an orthonormal basis over N_0^0, in the sense of 1.4, with $\eta_0 = 1$. Denote by $\{\xi_n\}_n \subset N$ the (countable) set of elements of the form $\xi_n = \otimes(\eta_{n_k})_k$, with $n_k \geq 0$ all but finitely many equal to 0, at least one being ≥ 1. It is immediate to see that $\{1\} \cup \{\xi_n\}_n$ is an orthonormal basis of $L^2 N$ over N^0 that checks condition (2.2.1) with respect to the Bernoulli $\Gamma \curvearrowright K$ action σ.

To prove 4°, note first that N_0 abelian diffuse implies $N_0 \simeq L^\infty(\mathbb{T}, \lambda)$, with τ_0 corresponding to $\int \cdot \mathrm{d}\mu$. To construct α and β it is then sufficient to construct an action $\alpha_0 : \mathbb{R} \to \mathrm{Aut}(A_0 \overline{\otimes} A_0, \tau_0 \otimes \tau_0)$ and a period 2 automorphism $\beta_0 \in \mathrm{Aut}(A_0 \overline{\otimes} A_0, \tau_0 \otimes \tau_0)$

such that $\alpha_0(A_0 \otimes 1) = 1 \otimes A_0$, $\beta_0(a_0 \otimes 1) = a_0 \otimes 1$, $\forall a_0 \in A_0$, $\beta_0\alpha_0(t) = \alpha_0(-t)\beta_0$, $\forall t \in \mathbb{R}$. Indeed, because then the product actions $\alpha_t = \otimes_k(\alpha_0(t))_k$, $\beta = \otimes_k(\beta_0)_k$ will clearly satisfy conditions 2.1.1, 2.1.2.

Since A_0 is diffuse, $(A_0, \tau_0) \simeq (L^\infty(\mathbb{T}, \mu), \int \cdot \mathrm{d}\mu)$. Let u, v be Haar unitaries generating $A_0 \otimes 1$ and $1 \otimes A_0$. Then u, v is a *pair of Haar unitaries* for $(\tilde{A}_0, \tilde{\tau}_0) = (A_0 \overline{\otimes} A_0, \tau_0 \otimes \tau_0)$, i.e. u, v generate $\tilde{A}_0$ and $\tilde{\tau}_0(u^n v^m) = 0$ for all $(n, m) \neq (0, 0)$, where $\tilde{\tau}_0 = \tau_0 \otimes \tau_0$. Note that if $u', v'\tilde{A}_0$ is any other pair of Haar unitaries, then there exists a unique automorphism θ of $(\tilde{A}_0, \tilde{\tau}_0)$ that takes u to u' and v to v', defined by $\theta(u^n v^m) = (u')^n (v')^m, \forall n, m \in \mathbb{Z}$. Note also that if $w \in \mathcal{U}(\tilde{A}_0)$ belongs to an algebra which is $\tilde{\tau}_0$-independent of the algebra generated by u then wu is a Haar unitary.

With this in mind, let h be the unique selfadjoint element in $\tilde{A}_0$ with spectrum in $[0, 2\pi]$ such that $e^{ih} = vu^*$. From the above remark it follows that both $u' = e^{ith}u$ and $v' = e^{ith}v$ are Haar unitaries in $\tilde{A}_0$. Also, $v'u'^* = vu^*$ so the von Neumann algebra generated by u', v' contains h, thus it also contains u, v, showing that u', v' generate $\tilde{A}_0$, thus being a pair of Haar unitaries for $\tilde{A}_0$. Thus, there exists a unique automorphism $\alpha_0(t)$ of $\tilde{A}_0$ such that $\alpha_0(t)(u) = e^{ith}u$, $\alpha_0(t)(v) = e^{ith}v$. By definition, we see that $\alpha_0(t)(vu^*) = vu^*$, thus $\alpha_0(t)(h) = h$ as well. But this implies that for all $t, t' \in \mathbb{R}$ we have

$$\alpha_0(t')(\alpha_0(t)(u)) = \alpha_0(t')(e^{ith}u) = e^{ith}\alpha_0(t')(u) = e^{ith}e^{it'h}u = \alpha_0(t' + t)(u).$$

Similarly $\alpha_0(t')(\alpha_0(t)(v)) = \alpha_0(t' + t)(v)$, showing that $\alpha_0(t' + t) = \alpha_0(t')\alpha_0(t)$, $\forall t, t'$.

Further on, since u^*, uv^* is a Haar pair for $\tilde{A}_0$, there exists a unique automorphism β_0 of $(\tilde{A}_0, \tilde{\tau}_0)$ such that $\beta_0(u) = u^*$, $\beta_0(vu^*) = vu^*$. By definition, β_0 satisfies $\beta_0^2(u) = u$ and $\beta_0^2(vu^*) = vu^*$, showing that $\beta_0^2 = id$. Finally, since $\beta_0(h) = h$, we have $\alpha_0(t)(\beta_0(u)) = e^{-ith}u^* = \beta_0(\alpha_0(-t)(u))$. Similarly $\alpha_0(t)(\beta_0(v)) = \beta_0(\alpha_0(-t)(v))$, altogether showing that $\alpha_0(t)\beta_0 = \beta_0\alpha_0(-t)$. Thus, σ is s-malleable.

Assume now that N_0 is a tensor product of 2 by 2 matrix algebras. Consider first the case $N_0 = M_{2\times 2}(\mathbb{C})$. We let $\{e_{ij}\}_{i,j=1,2}$ be a matrix unit for N_0. Define the unitaries $\alpha_0(t) \in N_0 \otimes N_0$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t/2 & \sin \pi t/2 & 0 \\ 0 & -\sin \pi t/2 & \cos \pi t/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This clearly defines a continuous unitary representation of $\mathbb{R}$. Then we define $\beta_0 \in \mathbb{C} \otimes N_0$ to be the unitary element:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

An easy calculation shows that $\beta_0\alpha_0(t)\beta_0^{-1} = \alpha_0(-t)$ and that $\mathrm{Ad}(\alpha_0(1))(N_0 \otimes \mathbb{C}) \perp N_0 \otimes 1$. By taking tensor products of the above construction, this shows that if $(N_0, \tau_0) = \overline{\otimes}_l (M_{2\times 2}(\mathbb{C}), tr)_l$ then there exists a continuous action $\alpha_0 : \mathbb{R} \to \mathrm{Aut}(N_0 \overline{\otimes} N_0, \tau_0 \otimes \tau_0)$ and a period 2-automorphism β_0 of N_0 such that $\alpha_0(1)(N_0 \otimes 1) \perp N_0 \otimes 1$, $\mathrm{sp}(N_0 \otimes 1)\alpha_0(1)(N_0 \otimes 1)$ dense in $N_0 \overline{\otimes} N_0$, $N_0 \otimes 1$ fixed by β, $\beta\alpha_0(t) = \alpha_0(-t)\beta, \forall t \in \mathbb{R}$.

But then $\alpha = \otimes_k (\alpha_0)_k$, $\beta = \otimes_k (\beta_0)_k$ clearly satisfy (4.3.1′), (4.3.2′). $\square$

4.6. Lemma. *Let Γ be a countable discrete group and $H \subset \Gamma$ a subgroup with the relative property* (T). *Let σ be a s-malleable action of Γ on the finite von Neumann algebra (P, τ), with $\sigma_{|H}$ weak mixing. Let $\tilde{\sigma}$, $\alpha : \mathbb{R} \to \mathrm{Aut}(P \overline{\otimes} P, \tau \otimes \tau)$, $\beta \in \mathrm{Aut}(P \overline{\otimes} P, \tau \otimes \tau)$ be as in Definition* 4.3. *Let (N, τ) be a finite von Neumann algebra and ρ and action of Γ on it. If w is a cocycle for the action $\sigma_g \otimes \rho_g$ of Γ on $P \overline{\otimes} N$ then $w_{|H}$ and $\alpha_1(w)_{|H}$ are equivalent as cocycles for the action $\tilde{\sigma}_h \otimes \rho_h$ of H on $P \overline{\otimes} P \overline{\otimes} N$.*

Proof. Denote $\tilde{P} = P \overline{\otimes} P$. We still denote by α, β the N-amplifications $\alpha \otimes id_N$ and $\beta \otimes id_N$. Also, denote $\sigma'_g = \sigma_g \otimes \rho_g$ and $\tilde{\sigma}'_g = \tilde{\sigma}_g \otimes \rho_g$. Since α is a continuous action by automorphisms commuting with $\tilde{\sigma}$, α acts continuously on the set of cocycles w for the action $\tilde{\sigma}'$ of Γ on $\tilde{P} \overline{\otimes} N$.

Note that by part 2° of Lemma 2.12 it is sufficient to prove that for any $\varepsilon > 0$ there exists $v_1 \in \tilde{P} \overline{\otimes} N$ partial isometry such that $\|v_1 - 1\|_2 \leq \varepsilon$ and $w_h \tilde{\sigma}'_h(v_1) = v_1 \alpha_1(w_h)$, $\forall h \in H$. Indeed, because if u is a unitary in $\tilde{P} \overline{\otimes} N$ such that $uv_1^* v_1 = v_1$ then

$$\|w_h \tilde{\sigma}'_h(u) - u\alpha_1(w_h)\|_2 \leq 2\|u - v_1\|_2 = 2\|1 - v_1^* v_1\|_2 \leq 4\varepsilon,$$

forall $h \in H$, and 2.12.2° applies.

To construct such v_1, note first that by Lemma 4.2 for the given $\varepsilon > 0$ there exists n such that if we denote $t_0 = 2^{-n}$ then there exists a partial isometry $v_0 \in \tilde{P} \overline{\otimes} N$ satisfying $w_h \tilde{\sigma}'_h(v_0) = v_0 \alpha_{t_0}(w_h)$, $\forall h \in H$, and $\|v_0 - 1\|_2 \leq \varepsilon$.

Let us now show that if

$$w_h \tilde{\sigma}'_h(v) = v\alpha_t(w_h), \forall h \in H, \tag{4.6.1}$$

for some $0 < t < 1$ and a partial isometry $v \in \tilde{P} \overline{\otimes} N$, then there exists a partial isometry $v' \in \tilde{P} \overline{\otimes} N$ satisfying $\|v'\|_2 = \|v\|_2$ and $w_h \tilde{\sigma}'_h(v') = v' \alpha_{2t}(w_h)$, $\forall h \in H$. This will of course prove the existence of v_1, by starting with $t = t_0 = 2^{-n}$ then proceeding by induction until we reach $t = 1$ (after n steps).

Applying β to (4.6.1) and using that β commutes with $\tilde{\sigma}'$, $\beta(x) = x, \forall x \in P \overline{\otimes} N \subset \tilde{P} \overline{\otimes} N$ and $\beta\alpha_t = \alpha_{-t}\beta$ as automorphisms on $\tilde{P} \overline{\otimes} N$, we get $\beta(w_h) = w_h$ and

$$w_h \tilde{\sigma}'_h(\beta(v)) = \beta(v)\alpha_{-t}(w_h), \forall h \in H. \tag{4.6.2}$$

Since (4.6.1) can be read as $v^*w_h = \alpha_t(w_h)\tilde{\sigma}'_h(v^*)$, from (4.6.1) and (4.6.2) we get the identity

$$v^*\beta(v)\alpha_{-t}(w_h) = v^*w_h\tilde{\sigma}'_h(\beta(v))$$

$$= \alpha_t(w_h)\tilde{\sigma}'_h(v^*)\tilde{\sigma}'_h(\beta(v)) = \alpha_t(w_h)\tilde{\sigma}'_h(v^*\beta(v)),$$

for all $h \in H$. By applying α_t on both sides of this equality, if we denote $v' = \alpha_t(\beta(v)^*v)$ then we further get

$$v'^*w_h = \alpha_{2t}(w_h)\tilde{\sigma}'_h(v'^*), \forall h \in H,$$

showing that $w_h\tilde{\sigma}'_h(v') = v'\alpha_{2t}(w_h)$, $\forall h \in H$, as desired. On the other hand, the intertwining relation (4.6.1) implies that vv^* is in the fixed point algebra B of the action $\mathrm{Ad}w_h \circ \tilde{\sigma}'_h$ of H on $\tilde{P}\overline{\otimes}N$. Since $\tilde{\sigma}'_{|H}$ is weak mixing on $(1 \otimes P) \otimes 1 \subset \tilde{P}\overline{\otimes}N$ (because it coincides with σ on $1_P \otimes P \otimes 1_N \simeq P$) and because $\mathrm{Ad}w_h$ acts as the identity on $(1 \otimes P) \otimes 1$ and leaves $(P \otimes 1)\overline{\otimes}N$ globally invariant, it follows that B is contained in $(P \otimes 1)\overline{\otimes}N$. Thus β acts as the identity on it (because it acts as the identity on both $P \otimes 1$ and $1 \otimes N$). In particular $\beta(vv^*) = vv^*$, showing that the right support of $\beta(v^*)$ equals the left support of v. Thus, $\beta(v^*)v$ is a partial isometry having the same right support as v, implying that v' is a partial isometry with $\|v'\|_2 = \|v\|_2$. $\square$

5. Cocycle and OE superrigidity results

5.1. Definitions. An infinite subgroup H of a group Γ is *w-normal* (resp. *wq-normal*) in Γ if there exists an ordinal $\imath$ and a well ordered family of intermediate subgroups $H = H_0 \subset H_1 \subset ... \subset H_\jmath \subset ... \subset H_\imath = \Gamma$ such that for each $0 < \jmath \leq \imath$, the group $H'_\jmath = \cup_{n<\jmath} H_n$ is normal in $H_\jmath$ (resp. $H_\jmath$ is generated by the elements $g \in \Gamma$ with $|gH'_\jmath g^{-1} \cap H'_\jmath| = \infty$).

For instance, if $H = H_0 \subset H_1 \subset ... \subset H_n = \Gamma$ are all normal inclusions and H is infinite then $H \subset \Gamma$ is w-normal. An inclusion of the form $H \subset (H * \Gamma_0) \times \Gamma_1$ with H, Γ_1 infinite and Γ_0 arbitrary is wq-normal but not w-normal.

All statements in this Section will be about inclusions of countable infinite groups $H \subset \Gamma$ with the relative property (T) of Kazhdan-Margulis (or $H \subset \Gamma$ is a rigid subgroup, i.e. it checks condition (4.1.2)) such that H is either wq-normal or w-normal in Γ. The class of groups having infinite wq-normal rigid subgroups is closed to finite index restriction/extension, to normal extensions and to operatios such that $\Gamma \mapsto (\Gamma * \Gamma_0) \times \Gamma_1$, where Γ_0 is arbitrary and $|\Gamma_1| = \infty$. The class of groups having infinite w-normal rigid subgroups is closed to normal extensions. Both classes are closed to inductive limits.

Any lattice Γ in a Lie group of real rank at least 2, such as $SL(n, \mathbb{Z})$, $n \geq 3$, has property (T) by Kazhdan's celebrated result ([K]), so $H = \Gamma \subset \Gamma$ is rigid. The normal

inclusions of groups $\mathbb{Z}^2 \subset \Gamma = \Gamma_0 \ltimes \mathbb{Z}^2$, with $\Gamma_0 \subset SL(2,\mathbb{Z})$ non amenable, are rigid (cf. [K], [M], [B]), and so are all inclusions $\mathbb{Z}^n \subset \Gamma = \Gamma_0 \ltimes \mathbb{Z}^n$ in ([Va], [Fe]). For each such $H \subset \Gamma$, the inclusion $H \subset \Gamma' = \Gamma \times H'$ is w-normal rigid, for any group H'. Also, if $H \subset \Gamma$ is wq-normal rigid then $H \subset (\Gamma * \Gamma_0) \times \Gamma_1$ is wq-normal rigid whenever Γ_1 is an infinite group, Γ_0 arbitrary, but it can be shown that if Γ_0 is non-trivial then $(\Gamma * \Gamma_0) \times \Gamma_1$ has no w-normal rigid subgroups.

Notice that in the cocycle superrigidity results 5.2-5.5 below the actions σ are not assumed to be free.

5.2. Theorem (Cocycle superrigidity of s-malleable actions). *Let σ be a s-malleable action of a countable discrete group Γ on the standard probability space (X,μ). Let $H \subset \Gamma$ be a rigid subgroup such that $\sigma_{|H}$ is weak mixing. Let $\mathcal{V} \in \mathscr{U}_{fin}$.*

If ρ is an arbitrary action of Γ on a standard probability space (Y,ν), then any $\mathcal{V}$-valued cocycle w for the diagonal product action $\sigma_g \times \rho_g, g \in \Gamma$, on $(X \times Y, \mu \times \nu)$ is cohomologous to a $\mathcal{V}$-valued cocycle w' whose restriction to H is independent on the X-variable. If in addition H is w-normal in Γ, or if H is wq-normal in Γ but σ is mixing, then w' is independent on the X-variable on all Γ, in other words the inclusion $\mathrm{Z}^1(\rho;\mathcal{V}) \subset \mathrm{Z}^1(\sigma \times \rho;\mathcal{V})$ implements an isomorphism between the sets of equivalence classes $\mathrm{Z}^1(\rho;\mathcal{V})/\sim$ and $\mathrm{Z}^1(\sigma\times\rho;\mathcal{V})/\sim$.

In particular, any cocycle w for σ with values in $\mathcal{V}$ is cohomologous to a cocycle w' which restricted to H is a group morphism of H into $\mathcal{V}$ and if H is w-normal in Γ, or if H is wq-normal in Γ but σ is mixing, then $\mathrm{Z}^1(\sigma;\mathcal{V}) = \mathrm{Z}^1_0(\sigma;\mathcal{V})$.

Proof. The last part follows by simply taking ρ to be the Γ-action on the one point probability space. To prove the first part, by Proposition 3.5 it is sufficient to consider the case $\mathcal{V} = \mathcal{U}(N_0)$, with N_0 a separable finite von Neumann algebra. But if $\mathcal{V} = \mathcal{U}(N_0)$ and σ s-malleable, then the statement follows from Lemma 4.6 and Theorem 3.1. $\square$

5.3. Theorem (Cocycle superrigidity of sub s-malleable actions). *The same statement as in 5.2 holds true if instead of σ s-malleable with $\sigma_{|H}$ weak mixing, we merely assume that σ is the quotient of a s-malleable action $\Gamma \curvearrowright^{\sigma'} (X',\mu')$ such that $\sigma'_{|H}$ is weak mixing on (X',μ') and weak mixing relative to $L^\infty X$.*

Proof. Trivial by 5.2 and 2.11. $\square$

5.4. Corollary (Cocycle superrigidity of generalized Bernoulli actions). *Let σ be a (X_0,μ_0)-Bernoulli $\Gamma \curvearrowright K$ action, where Γ is a countable discrete group acting on the countable set K and (X_0,μ_0) is a non-trivial standard probability space. Let $H \subset \Gamma$ be an infinite subgroup with the relative property* (T) *such that all orbits of $H \curvearrowright K$ are infinite (see* 4.5.2°*). Let $\mathcal{V}$ be a Polish group of finite type.*

Then any cocycle $w : X \times \Gamma \to \mathcal{V}$ for σ with values in $\mathcal{V}$ is cohomologous to a cocycle w' whose restriction to H is a group morphism of H into $\mathcal{V}$. If in addition H

is w-normal in Γ*, or if* H *is wq-normal but* $\{g \in \Gamma \mid gk = k\}$ *is finite,* $\forall k \in K$ *(see* $4.5.3°$*), then* w' *follows a group morphism on all* Γ*.*

More generally, if ρ *is an arbitrary action of* Γ *on a standard probability space* (Y, ν) *and* w *is a cocycle for the diagonal product action* $\sigma_g \times \rho_g$*,* $g \in \Gamma$*, on* $(X \times Y, \mu \times \nu)$*, then any* $\mathcal{V}$*-valued cocycle* w *for* $\sigma \times \rho$ *is cohomologous to a* $\mathcal{V}$*-valued cocycle* w' *whose restriction to* H *is independent on the* X*-variable, i.e. to a cocycle for* $\rho_{|H}$*. Moreover, if* H *is w-normal in* Γ*, or if* H *is wq-normal but* $\{g \in \Gamma \mid gk = k\}$ *finite,* $\forall k \in K$*, then* w' *follows independent on the* X*-variable on all* Γ*.*

5.5. Theorem (Cocycle superrigidity: the non-commutative case). *Let* σ *be a s-malleable action of a countable discrete group* Γ *on a finite von Neumann algebra* (P, τ)*. Let* $H \subset \Gamma$ *be a rigid subgroup such that* $\sigma_{|H}$ *is weak mixing. Let* (N, τ) *be an arbitrary finite von Neumann algebra and* ρ *an action of* Γ *on* (N, τ)*. Then any cocycle* w *for the diagonal product action* $\sigma \otimes \rho$ *of* Γ *on* $P \overline{\otimes} N$ *is equivalent to a cocycle* w' *whose restriction to* H *takes values in* $N = 1 \otimes N$*. If in addition* H *is w-normal in* Γ*, or if* H *is wq-normal in* Γ *but* σ *is mixing, then* w' *takes values in* N *on all* Γ*.*

Proof. Let $\sigma'_g = \sigma_g \otimes \rho_g$, $g \in \Gamma$. Let $\tilde{\sigma} : \Gamma \to \mathrm{Aut}(\tilde{P}, \tilde{\tau})$ denote the diagonal product action $\tilde{\sigma}_g = \sigma_g \otimes \sigma_g$ of Γ on $\tilde{P} = P \overline{\otimes} P$ and $\alpha : \mathbb{R} \to \mathrm{Aut}(\tilde{P}, \tilde{\tau})$, $\beta \in \mathrm{Aut}(\tilde{P}, \tilde{\tau})$ as in Definition 4.3. By Lemma 4.6, $w_{|H} \sim \alpha_1(w)_{|H}$ as cocycles for $\tilde{\sigma}_h \otimes \rho_h$, $h \in H$. But then Proposition 3.2 implies $w_{|H}$ is equivalent to a cocycle w' with $w'_h \in \mathcal{U}(N)$, $\forall h \in H$. $\square$

We'll now apply the case $\mathcal{V} = \Lambda$ discrete of the Cocycle Superrigidity 5.3 to deduce orbit equivalence superrigidity of the corresponding source actions. The results will in fact hold true for all source actions that are cocycle superrigid, in the following sense:

5.6.0. Terminology. Let $\mathscr{U}$ be a family of Polish groups. We say that $\Gamma \curvearrowright^{\sigma} (X, \mu)$ is *$\mathscr{U}$-cocycle superrigid* if any $\mathcal{V}$-valued cocycle for σ is cohomologous to a group morphism of Γ into $\mathcal{V}$, $\forall \mathcal{V} \in \mathscr{U}$. If $\mathscr{U}$ is the family of all discrete groups, we simply say that σ is *cocycle superrigid*. With this terminology, Theorem 5.2 shows that if Γ has an infinite subgroup $H \subset \Gamma$ with the relative property (T), σ is a weak mixing s-malleable Γ-action and either H is w-normal in Γ, with $\sigma_{|H}$ weak mixing, or H is wq-normal, with σ mixing, then σ is cocycle superrigid. By 5.3, for an action $\Gamma \curvearrowright^{\sigma} X$ to be cocycle superrigid it is in fact sufficient that it is the quotient of a cocycle superrigid action $\Gamma \curvearrowright^{\sigma'} X'$ with the property that $\Gamma \curvearrowright L^\infty X'$ is weak mixing relative to $L^\infty X$ (in the sense of Definition 2.9).

5.6.1. Assumption. $\Gamma \curvearrowright^{\sigma} X$ is free and, as an action on $L^\infty X$, it admits an extension to a s-malleable action σ' of Γ on a larger abelian von Neumann algebra (A', τ') with the property that $\tau'_{|A} = \tau$, σ' weak mixing relative to $L^\infty X \subset A'$ and such that there exists an infinite rigid subgroup $H \subset \Gamma$ with $\sigma'_{|H}$ weak mixing on A' and either H is w-normal in Γ or H is merely wq-normal but σ' mixing.

Examples of (σ, Γ) satisfying 5.6.1 are all (X_0, μ_0)-Bernoulli $\Gamma \curvearrowright K$ actions of groups

Γ having an infinite rigid subgroup $H \subset \Gamma$ with the property that $\Gamma \curvearrowright K$ satisfies 4.5.1° and either $H \curvearrowright K$ checks 4.5.2° and $H \subset \Gamma$ is w-normal, or Γ checks 4.5.3° and H is wq-normal.

Recall from 1.4 that if $\mathcal{R}$, $\mathcal{S}$ are countable m.p. relations on the probability spaces $(X, \mu), (Y, \nu)$ then an *orbit equivalence* (*OE*) of $\mathcal{R}$, $\mathcal{S}$ is an isomorphism of probability spaces $\Delta : (X, \mu) \simeq (Y, \nu)$ which takes $\mathcal{R}$ onto $\mathcal{S}$. A *local OE* of $\mathcal{R}$, $\mathcal{S}$ is a m.p. map $\Delta : (X, \mu) \to (Y, \nu)$ (a.e. surjective, but not necessarily a.e. 1 to 1) with the property that for almost all $t \in X$, Δ gives a bijection between the $\mathcal{R}$-orbit of t and the $\mathcal{S}$-orbit of $\Delta(t)$ (see 1.4.2). For instance, if $\Gamma \curvearrowright^{\sigma} (X, \mu)$ is a free m.p action, $B \subset L^\infty X$ is a $\sigma(\Gamma)$-invariant von Neumann subalgebra of $L^\infty X$ such that $\sigma_{|B}$ is still free and we write $B = L^\infty Y$, for some probability space (Y, ν) with the property that τ_μ equals τ_ν on B, then the associated quotient map $\Delta : (X, \mu) \to (Y, \nu)$ (see 1.1.1) implements a local OE of $\mathcal{R}_\sigma$, $\mathcal{R}_\theta$, where θ_g is the action of Γ on (Y, μ) implemented by the restriction of σ_g to $L^\infty Y = B$. An *embedding* of $\mathcal{R}$ into $\mathcal{S}$ is an isomorphism $\Delta : (X, \mu) \simeq (Y, \nu)$ that takes $\mathcal{R}$ onto a subequivalence relation of $\mathcal{S}$.

5.6. Theorem (OE Superrigidity). *Let $\Gamma \curvearrowright^{\sigma} X$ be a free, weakly mixing, cocycle superrigid action, e.g. an action satisfying* 5.6.1*. Assume Γ has no finite normal subgroups. Let θ be an arbitrary free ergodic m.p. action of a countable discrete group Λ on a standard probability space (Y, ν) and $\Delta : \mathcal{R}_\sigma \simeq \mathcal{R}_\theta^t$ an orbit equivalence, for some $t > 0$.*

Then $n = 1/t$ is an integer and there exist a subgroup $\Lambda_0 \subset \Lambda$ of index $[\Lambda : \Lambda_0] = n$, a subset $Y_0 \subset Y$ of measure $\nu(Y_0) = 1/n$ fixed by $\theta_{|\Lambda_0}$, an automorphism $\alpha \in [\mathcal{R}_\theta]$ and a group isomorphism $\delta : \Gamma \simeq \Lambda_0$ such that $\alpha \circ \Delta$ takes X onto Y_0 and conjugates the actions $\sigma, \theta_0 \circ \delta$, where θ_0 denotes the action of Λ_0 on Y_0 implemented by θ.

Moreover, if Γ is assumed ICC but the free action $\Gamma \curvearrowright^{\sigma} X$ is merely a quotient of a weakly mixing cocycle superrigid action (e.g. of an action satisfying 5.6.1*), then $\Gamma \curvearrowright X$ is OE superrigid, i.e. if $\Lambda \curvearrowright Y$ is a free ergodic m.p. action of an arbitrary countable group Λ and $\Delta : X \simeq Y$ is an orbit equivalence of $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ then there exist $\alpha \in [\Lambda]$ and $\delta : \Gamma \simeq \Lambda$ such that $\Delta_0 = \alpha \circ \Delta$ conjugates $\sigma, \theta \circ \delta$.*

5.7. Theorem (Superrigidity of local OE). *Let $\Gamma \curvearrowright^{\sigma} (X, \mu)$ be a free weakly mixing cocycle superrigid action, e.g. an action satisfying* 5.6.1*. Assume Γ has no finite normal subgroups. Let θ be an arbitrary free ergodic m.p. action of a countable discrete group Λ on a standard probability space (Y, ν). Let $\Delta : (X, \mu) \to (Y, \nu)^t$ be a local OE of $\mathcal{R}_\sigma$, $\mathcal{R}_\theta^t$, for some $t > 0$.*

Then $n = t^{-1}$ is an integer and if θ' denotes the pull back of θ to a Λ-action on (X^n, μ_{X^n}) (see 1.4*) then there exist a subgroup $\Lambda_0 \subset \Lambda$ of index $[\Lambda : \Lambda_0] = n$, a subset $X_0 \subset X^n$ of measure $\mu_{X^n}(X_0) = 1/n$ fixed by $\theta'_{|\Lambda_0}$, an automorphism $\alpha \in [\mathcal{R}_{\sigma^n}]$ and a group isomorphism $\delta : \Gamma \simeq \Lambda_0$ such that α takes X onto X_0 and conjugates the actions $\sigma, \theta_0 \circ \delta$, where θ_0 denotes the action of Λ_0 on X_0 implemented by θ'.*

In particular, if Δ is a local OE of $\mathcal{R}_\sigma, \mathcal{R}_\theta$ then there exist $\alpha \in [\mathcal{R}_\sigma]$ and $\delta : \Gamma \simeq \Lambda$ such that $\theta \circ \delta$ is a quotient of $\alpha\sigma\alpha^{-1}$ (via Δ).

5.8. Theorem (Superrigidity of embeddings). *Let $\Gamma \curvearrowright^\sigma (X, \mu)$ be a free weakly mixing cocycle superrigid action, e.g. an action satisfying 5.6.1. Assume Γ has no finite normal subgroups. Let $\Lambda \curvearrowright^\theta (Y, \nu)$ be an arbitrary action and $\Delta : (X, \mu) \simeq (Y, \nu)^t$ an embedding of $\mathcal{R}_\sigma$ into $\mathcal{R}_\theta^t$, for some $t > 0$, such that any Γ-invariant finite subequivalence relation of $\mathcal{R}_\theta^t$ is contained in $\mathcal{R}_\sigma$ (when identifying $\mathcal{R}_\sigma$ with a subequivalence of $\mathcal{R}_\theta^t$, via Δ).*

Then $t \leq 1$ and there exist an isomorphism δ of Γ onto a subgroup Λ_0 of Λ and $\alpha \in [\mathcal{R}_\theta]$ such that $\Delta_0 = \alpha \circ \Delta$ takes X onto a Λ_0-invariant subset $Y_0 \subset Y$, with $\mu(Y_0) = t$, and conjugates the action σ with the action $\theta_{|\Lambda_0}$ of Λ_0 on Y_0, with respect to the isomorphism $\delta : \Gamma \simeq \Lambda_0$.

5.9. Corollary. *Let $\Lambda \curvearrowright (X, \mu)$ be a free ergodic m.p. action and assume there exists $t > 0$ such that $\mathcal{R}_\Lambda^t \supset \mathcal{R}_\Gamma$ for some $\Gamma \curvearrowright (X, \mu)$ satisfying 5.6.1, Γ without finite normal subgroups and such that any Γ-invariant finite subequivalence relation of $\mathcal{R}_\Lambda^t$ is contained in $\mathcal{R}_\Gamma$. Then $\mathscr{F}(\mathcal{R}_\Lambda) = \{1\}$. Thus, if $\mathcal{R}$ is a countable m.p. equivalence relation with $\mathscr{F}(\mathcal{R}) \neq \{1\}$ and such that $\mathcal{R} \supset \mathcal{R}_\Gamma$ for some $\Gamma \curvearrowright X$ satisfying the above properties then $\mathcal{R}^t$ cannot be implemented by a free group action, $\forall t > 0$.*

5.10. Corollary. *Let Γ be a group having an infinite wq-normal rigid subgroup and no finite normal subgroup. Let $(X, \mu) = (\{0, 1\}, \mu_0)^\Gamma$, with $s = \mu_0(\{0\})/\mu_0(\{1\}) \neq 1$. Let $\mathcal{R}_0$ be the hyperfinite equivalence relation on $\{0, 1\}^\Gamma$ given by: $(t_g)_g \sim (t'_g)_g$ iff there exists a finite subset $F \subset \Gamma$ such that $t_g = t'_g$, $\forall g \in \Gamma \setminus F$ and $\Pi_{g \in F}\mu_0(t_g) = \Pi_{g \in F}\mu_0(t'_g)$. Let $\mathcal{R}$ be the countable m.p. equivalence relation generated by $\mathcal{R}_0$ and the Bernoulli Γ-action $\Gamma \curvearrowright X$ (which leaves $\mathcal{R}_0$ invariant). Then $\mathcal{R}^t$ cannot be implemented by a free group action, $\forall t > 0$. Moreover, $\mathscr{F}(\mathcal{R}) \supset s^{\mathbb{Z}}$ and if Γ is of the form $\Gamma_0 \ltimes \mathbb{Z}^2$, with Γ_0 a finitely generated non virtually cyclic subgroup of $SL(2, \mathbb{Z})$, then $\mathscr{F}(\mathcal{R}) = s^{\mathbb{Z}}$.*

We deduce Theorems 5.6-5.8 and their Corollaries 5.9, 5.10 from the Cocycle Superrigidity 5.2-5.3 and a *General Principle* (Proposition 5.11 below) showing that any "untwister" of a Zimmer cocycle associated with an orbit equivalence of actions (defined below) gives rise to a natural "conjugator" of the two actions. We in fact prove a more general such principle, dealing also with embeddings and local OE of equivalence relations, and their amplifications. Prior results along this line have been obtained in (4.2.9, 4.2.11 of [Z2]) and (3.3 of [Fu2], 2.4 of [Fu3]). The proof uses von Neumann algebra framework, but is otherwise quite straightforward.

To formulate this result, let us recall the definition of the *Zimmer cocycle* w_Δ associated with a morphism Δ of equivalence relations implemented by group actions, with the target action free. Thus, let σ be a free Γ-action on (X, μ) and θ a free Λ-action on (Y, ν). Let $Y_0 \subset Y$ be a subset of positive measure and assume $\Delta : (X, \mu) \to (Y_0, \nu_{Y_0})$ is

a m.p. morphism of $\mathcal{R}_\sigma$ into $\mathcal{R}_\theta^{Y_0}$, with $N_0 \subset X$ a subset of measure 0 such that Δ takes the $\mathcal{R}_\sigma$-orbit of any $t \in X \setminus N_0$ into the $\mathcal{R}_\theta^{Y_0}$-orbit of $\Delta(t)$. Let $w = w_\Delta : X \times \Gamma \to \Lambda$ be defined as follows: For given $t \in X \setminus N_0$, $g \in \Gamma$, $w(t, g)$ is the unique $h \in \Lambda$ such that $\Delta(\sigma_g^{-1}(t)) = \theta_h^{-1}(\Delta(t))$. It is immediate to check that $w : X \times \Gamma \to \Lambda$ this way defined is measurable and satisfies the (right) cocycle relation (2.1.1) (see [Z2]).

The "principle" below shows that if $\Gamma \curvearrowright^\sigma (X, \mu)$ is free and weak mixing, $\Lambda \curvearrowright^\theta (Y, \nu)$ is free and Δ is a (local) OE (resp. an embedding) of the corresponding equivalence relations $\mathcal{R}_\sigma, \mathcal{R}_\theta$, then a measurable map $v : X \to \Lambda$ implementing an equivalence between w_Δ and a group morphism $\delta : \Gamma \to \Lambda$ gives rise to a natural automorphism $\alpha = \alpha_v$ in the full group $[\mathcal{R}_\sigma]$ (resp. $[\mathcal{R}_\theta]$) that (virtually) conjugates the Γ-actions $\Delta\sigma\Delta^{-1}, \theta \circ \delta$.

5.11. Proposition. *Let $\Gamma \curvearrowright^\sigma (X, \mu)$, $\Lambda \curvearrowright^\theta (Y, \nu)$ be free m.p. actions, with σ weak mixing, and $\Delta : (X, \mu) \to (Y_0, \nu_{Y_0})$ either an embedding or a local OE of $\mathcal{R}_\sigma, \mathcal{R}_\theta^{Y_0}$, for some $Y_0 \subset Y$. Denote $A = L^\infty X$, $B = L^\infty Y$, $M = A \rtimes \Gamma$, $P = B \rtimes \Lambda$, $p = \chi_{Y_0}$, and $u_g \in M$, $v_h \in P$ the canonical unitaries implementing σ, θ.*

Let $w = w_\Delta$ denote the Zimmer cocycle associated to Δ. Assume $\delta : \Gamma \to \Lambda$ is a group morphism and $v : X \to \Lambda$ a measurable map such that for each $g \in \Gamma$ we have $v(t)^{-1} w(t, g) v(g^{-1}t) = \delta(g)$, $\forall t \in X$ (a.e.). Let $\{X_h\}_h$ be the partition of X into measurable subsets such that $v(t) = h$ for $t \in X_h$, $h \in \Lambda$. Denote $q_h = \chi_{X_h} \in L^\infty X$, $h \in \Lambda$. Let $b \overset{\text{def}}{=} \Sigma_h q_h v_h$, which is viewed as an element in $L^2 P$ when Δ is an embedding, via the inclusion $M \subset P$ (see 1.4), and is viewed as an element in $L^2 M^n$ when Δ is a local OE, via the inclusion $P \subset M^n$ (see 1.4.3), where $n = \nu(Y_0)^{-1}$.

Then $u_g b = b v_{\delta(g)}, \forall g \in \Gamma$, b is a scalar multiple of a partial isometry, $K = ker(\delta)$ is a finite normal subgroup of Γ and we have:

(i). If Δ is a local OE then $bb^* = \Sigma_{k \in K} u_k$, $b^*b = |K| q$ with $q \in A^n$ a projection invariant to the pull back $\theta'_h, h \in \Lambda_0 = \delta(\Gamma)$, and $n = [\Lambda : \Lambda_0]$ is an integer. Moreover, if we denote $A^K = L^\infty(X/K)$ the fixed point subalgebra of A under the action $\sigma_{|K}$, then $\alpha = |K|^{-1} \mathrm{Ad}(b^*)$ implements an isomorphism of A^K onto $A^n q$ which conjugates the Γ/K-action $\sigma_0 = \sigma_{|A^K}$ and the Λ_0-action $\theta'_{|\Lambda_0}$ with respect to the identification $\Gamma/K \simeq \Lambda_0$ implemented by δ. In particular, if Γ has no finite normal subgroups then $b^*b = q, bb^* = p$, $\alpha = \mathrm{Ad} b^*$ has graph in $\mathcal{R}_\sigma^n = \mathcal{R}_{\theta'}$ and conjugates $\sigma, \theta'_{|\Lambda_0}$, with respect to the isomorphism $\delta : \Gamma \simeq \Lambda_0 \subset \Lambda$.

(ii). If Δ is an embedding and we identify $A^n = B$, $M^n \subset P$ and $\mathcal{R}_\sigma \subset \mathcal{R}_\theta^X$ via Δ, then there exists a finite $\sigma(\Gamma)$-invariant subequivalence relation $\mathcal{K} \subset \mathcal{R}_\theta^X$ containing $\mathcal{R}_{\sigma(K)}$, such that if we denote by m the cardinality of the orbits of $\mathcal{K}$, $A^{\mathcal{K}} \simeq L^\infty(X/\mathcal{K})$ the quotient (or fixed point algebra) under $\mathcal{K}$ then $\alpha = m^{-1} \mathrm{Ad}(b^*)$ implements an isomorphism of $A^{\mathcal{K}}$ onto $A^n q$ which conjugates the Γ/K-action $\sigma_0 = \sigma_{|A^{\mathcal{K}}}$ onto the Λ_0-action θ_0 implemented by $\theta_{|\Lambda_0}$ on $Bq = A^n q$, with respect to the identification $\Gamma/K \simeq \Lambda_0$ implemented by δ. If in addition any finite $\sigma(\Gamma)-$invariant subequivalence

relation of $\mathcal{R}_\theta^X$ *is contained in* $\mathcal{R}_\sigma$ *then* $\mathcal{K} = \mathcal{R}_{\sigma(K)}$, $b^*b = |K|q$, $bb^* = \Sigma_{k\in K}u_k$, $\alpha = |K|^{-1}\mathrm{Ad}(b^*)$ *and* α *implements an isomorphism of* A^K *onto* A^nq. *Also, if* Γ *has no non-trivial finite normal subgroups, then* b *is a partial isometry in* P *normalizing* $A^n = B$ *with left support* $p = \chi_X$ *that conjugates the* Γ*-action* σ *and the* Λ_0*-action* θ_0.

Let us first prove 5.6-5.10 assuming Proposition 5.11, then we prove the latter.

Proof of Theorem 5.6. The first part is trivial by Theorem 5.7. To prove the last part, assume $\Gamma \curvearrowright^\sigma X$ is a quotient of a free weakly mixing cocycle superrigid action $\Gamma \curvearrowright^{\sigma'} (X', \mu')$. Let $A' = L^\infty(X', \mu')$, $A = L^\infty X \subset A'$, $M = A' \rtimes \Gamma$ and $N = A \rtimes \Gamma \subset M$, with $\{u_g\}_g \in M$ the canonical unitaries implementing σ' on A' and σ on $A \subset A'$. Let $\{v_h\}_{h\in\Lambda} \subset N$ be the canonical unitaries implementing θ on $L^\infty Y = L^\infty X$ (identification via Δ). By Proposition 5.11, there exists a unitary element $u' = \Sigma_h q_h v_h \in M$ normalizing A' such that $u'u_g = v_{\delta(g)}u', \forall g \in \Gamma$. Letting u be the expectation $E_N(u') = \Sigma_h E_A(q_h)v_h \neq 0$ of u' on N, since $u_g, v_h \in N$, it follows that we still have $uu_g = v_{\delta(g)}u, \forall g \in \Gamma$. Thus $u'^*u \in \{u_g\}_g' \cap M$. But σ' weak mixing implies $\{u_g\}_g' \cap M \subset L\Gamma$ and if in addition Γ is ICC then $\{u_g\}_g' \cap M \subset L\Gamma = L\Gamma' \cap L\Gamma = \mathbb{C}$. Thus $\Sigma_h E_A(q_h)v_h \in \mathbb{C}u'$, implying that $E_A(q_h) = q_h, \forall h$, i.e. $u' \in N$. But then $\alpha = \mathrm{Ad}u'$ will do. $\square$

Proof of Theorem 5.7. Let m be the smallest integer ≥ 1 such that $m \geq t$. Let $\tilde\Lambda = \Lambda \times (\mathbb{Z}/m\mathbb{Z})$ and denote by $\tilde\theta$ the action of $\tilde\Lambda$ on $\tilde Y = Y \times (\mathbb{Z}/m\mathbb{Z})$ given by the product of the actions θ and the translations by elements in $\mathbb{Z}/m\mathbb{Z}$. Also, denote $s = m/t$ and $\tilde\theta'$ the Δ^s-pull back of $\tilde\theta$, where $\Delta^s : (X, \mu)^s \simeq (\tilde Y, \tilde\nu)$ is the s-amplification of Δ. Thus, we may regard $\mathcal{R}_\theta^t$ as $\mathcal{R}_{\tilde\theta}^{Y_0}$, for some subset $Y_0 \subset \tilde Y$ of measure t/m, and Δ^s as a local OE of $\mathcal{R}_\sigma^s = \mathcal{R}_{\tilde\theta'}, \mathcal{R}_{\tilde\theta}$. By Proposition 5.11$(i)$, it follows that m/t is an integer. By the choice of m this implies either $t \leq 1$ (and so $m = 1$) or $t = m$ is an integer ≥ 2. The second case implies $\mathcal{R}_\sigma = \mathcal{R}_{\tilde\theta'}$, and since σ is cocycle superrigid by 5.11(i) the actions $\sigma, \tilde\theta'$ are conjugate. But σ is weak mixing and $\tilde\theta'$ is not, contradiction. Thus $t \leq 1$ and the statement follows now trivially from the cocycle superrigidity of σ and 5.11(i). $\square$

Proof of Theorem 5.8. Let m be the smallest integer ≥ 1 such that $m \geq t$. Let $\tilde\Lambda = \Lambda \times (\mathbb{Z}/m\mathbb{Z})$ and denote by $\tilde\theta$ the action of $\tilde\Lambda$ on $\tilde Y = Y \times (\mathbb{Z}/m\mathbb{Z})$ given by the product of the actions θ and the translations by elements in $\mathbb{Z}/m\mathbb{Z}$. Thus, we may regard $\mathcal{R}_\theta^t$ as $\mathcal{R}_{\tilde\theta}^{Y_0}$, for some subset $Y_0 \subset \tilde Y$ of measure t/m. Let $s = m/t$. Thus, we may apply 5.11(ii) to get a subset $Y_0' \subset \tilde Y$, with $\mu(Y_0') = \mu(Y_0) = t/m$, a subgroup $\Lambda_0 \subset \tilde\Lambda$, such that $\tilde\theta_{|\Lambda_0}$ leaves Y_0' invariant, and $\alpha \in [\mathcal{R}_{\tilde\theta}]$ that takes $Y = \Delta(X)$ onto Y_0', such that $\alpha \circ \Delta$ conjugates $\sigma, \theta_0' \circ \delta$, where θ_0 is the action implemented by $\tilde\theta_{|\Lambda_0}$ on Y_0'. In particular θ_0 is weak mixing (because σ is). This implies that if we denote $q = \chi_{Y_0'}$ then the finite dimensional subspace $L^\infty(\mathbb{Z}/m\mathbb{Z})q$ of $L^\infty Yq$, which is clearly invariant to θ_0, must be reduced to the scalars. Thus $\tau(q) \leq 1/m$, in other words $t \leq 1$.

□

Proof of Corollary 5.9. If $\mathscr{F}(\mathcal{R}_\Lambda) \neq \{1\}$ then after amplifying $\mathcal{R}_\Lambda$ by a sufficiently large number we may assume $\mathcal{R}_\Gamma \subset \mathcal{R}_\Lambda^t$, for some $t > 1$. But this contradicts 5.8. □

Proof of Corollary 5.10. By ([P2]) we have $\mathscr{F}(\mathcal{R}) \supset s^{\mathbb{Z}}$, with equality whenever Γ is of the required form. If $\mathcal{R}^t = \mathcal{R}_\Lambda$ for some free ergodic Λ-action then $(\mathcal{R}_\Lambda)^{1/t} = \mathcal{R} \supset \mathcal{R}_\Gamma$, where $\mathcal{R}_\Gamma$ is the equivalence relation given by the Bernoulli Γ-action $\Gamma \curvearrowright X = X_0^\Gamma$. Thus, if we can show that any finite Γ-invariant subequivalence relation $\mathcal{R}' \subset \mathcal{R}$ is included into $\mathcal{R}_\Gamma$ then the rest of the statement follows from Corollary 5.9. With the notations in Sec. 1.4, let $M = L(\mathcal{R})$ be the von Neumann algebra of $\mathcal{R}$ with $u_g \in M, g \in \Gamma$, the canonical unitaries implementing $\Gamma \curvearrowright X$. Then $\mathcal{R}'$ Γ-invariant is equivalent to $B = L(\mathcal{R}')$ being $\mathrm{Ad}u_g$-invariant, $g \in \Gamma$. Note that $A = L^\infty X$ is contained in $L(\mathcal{R}')$ and $\mathcal{R}'$ is finite iff there exist finitely many partial isometries $\{v_i\}_i \in B$ normalizing A and such that $E_A(v_i^* v_j) = 0$ for $i \neq j$, $B = \Sigma_i v_i A$. If we denote $N = L(\mathcal{R}_\Gamma) = A \vee \{u_g\}_g$, this also implies that $\mathcal{H}_0 = \Sigma_i v_i N$ is a N-bimodule. If $\mathcal{R}' \not\subset \mathcal{R}_\Gamma$, then there exists i such that $v_i \notin N$. It follows that there exists a projection $0 \neq q \in A v_i^* v_i$ such that $v_i q = w u_h \in B$, for some $h \in \Gamma$ and $w \in L(\mathcal{R}_0)$ normalizing A with $E_A(w) = 0$. But the action $\sigma_g(x) = u_g x u_g^*$, $x \in L(\mathcal{R}_0)$, $g \in \Gamma$, is mixing (see e.g. 2.4.3 in [P1], or 1.6.2 in [P2]). Thus, $\forall \delta > 0$, $\exists g \in \Gamma$ such that $\|E_N(\sigma_g(w^*)v_j)\|_2 \leq \delta$, $\forall j$. Since $w = v_i q u_g^* \in \mathcal{H}_0$ and δ is arbitrary, this shows that $w = 0$, contradiction. □

Proof of Proposition 5.11. For each $g \in \Gamma$, $h \in \Lambda$, we denote $X_h^g = \{t \in X \mid w(t,g) = h\}$. By the definition of w it follows that for each $g \in \Gamma$, $\{X_h^g\}_h$ gives a (a.e.) partition of X into measurable subsets. Also, if σ is free, then both in the case Δ is an embedding or local OE, the sets $\{X_h^g\}_g$ are disjoint. Indeed, this is because in both cases the map Δ is 1 to 1 on (μ-almost) each orbit of $\mathcal{R}_\sigma$, so that if $t \in X_h^g \cap X_h^{g'}$ then $\Delta(g^{-1}t) = h^{-1}\Delta(t) = \Delta({g'}^{-1}t)$, implying that $g^{-1}t = {g'}^{-1}t$, thus $g = g'$ by the freeness of σ. Denote $p_h^g = \chi_{X_h^g} \in L^\infty X$.

We begin by re-writing the equivalence of the Λ-valued cocycles w, δ for the Γ-action σ in von Neumann algebra framework, by using the "dictionary" in Section 2. Thus, we view Λ as the (discrete) group of canonical unitaries $\{v_h^0 \mid h \in \Lambda\}$ of the group von Neumann algebra $L\Lambda$ associated with Λ and $w_g = w(\cdot, g) \in \Lambda^X \subset L^\infty X \overline{\otimes} L\Lambda$. Consequently, $w_g = \Sigma_h p_h^g \otimes v_h^0$ and $v = \Sigma_h q_h \otimes v_h^0$. Note that $\Sigma_h q_h = p$ and $\Sigma_h p_h^g = p, \forall g \in \Gamma$.

If we still denote by σ the action of Γ on $L^\infty X \overline{\otimes} L\Gamma$ given by $\sigma_g \otimes id, g \in \Gamma$, then the cocycle relation for w becomes $w_{g_1}\sigma_{g_1}(w_{g_2}) = w_{g_1 g_2}$, $\forall g_1, g_2 \in \Gamma$. Also, the equivalence $w \sim \delta$ given by v becomes

$$w_g \sigma_g(v) = v(1 \otimes v_{\delta(g)}^0), g \in \Gamma,$$

which in turn translates into the identities:

$$\Sigma_h p_h^g \sigma_g(q_{h^{-1}k}) = q_{k\delta(g)^{-1}}, \forall k \in \Lambda, g \in \Gamma. \tag{5.11.1}$$

We first prove that the set of conditions (5.11.1) is equivalent to the following intertwining relation, viewed in P when Δ is an embedding, respectively in M^n when Δ is a local OE:

$$u_g b = b v_{\delta(g)}, g \in \Gamma. \tag{5.11.2}$$

By using the definitions and the fact that $p_h^g v_h = p_h^g u_g$, (cf. 1.4), we get in P (resp. in M^n) the equalities

$$u_g b = (\Sigma_{h_1} p_{h_1}^g v_{h_1})(\Sigma_{h_2} q_{h_2} v_{h_2}) \tag{5.11.3}$$

$$= \Sigma_{h_1,h_2} p_{h_1}^g \sigma_g(q_{h_2}) v_{h_1 h_2} = \Sigma_k (\Sigma_h p_h^g \sigma_g(q_{h^{-1}k})) v_k$$

and also

$$b v_{\delta(g)} = \Sigma_k q_{k\delta(g)^{-1}} v_k. \tag{5.11.4}$$

Thus, (5.11.2) holds true if and only if we have the identities

$$\Sigma_h p_h^g \sigma_g(q_{h^{-1}k}) = q_{k\delta(g)^{-1}}, \forall k \in \Lambda, g \in \Gamma, \tag{5.11.5}$$

which are exactly the same as conditions (5.11.1). We have thus shown that b intertwines the representations $\{u_g\}_g, \{v_{\delta(g)}\}_g$ of Γ, i.e. $u_g b = b v_{\delta(g)}, \forall g \in \Gamma$. In particular, bb^* commutes with $\{u_g\}_g$ and b^*b commutes with $\{v_{\delta(g)}\}_g$. To see from this that $ker(\delta)$ is finite note that for each $g \in ker(\delta) \lhd \Gamma$ we have $u_g b = b v_e = b$, hence $u_g s = s$, where s is the support projection of bb^*. This shows that the left regular representation of ker(δ) contains the trivial representation of $ker(\delta)$, implying that $ker(\delta)$ is finite (see e.g. [D]).

Note now that if $q' \in B$ (resp. $q' \in A^n$) is a projection fixed by $\{v_{\delta(g)} \mid g \in \Gamma\}$ then $bq' = \Sigma_h q_h \theta_h(q') v_h$ is still an intertwiner between $\{u_g\}_g$ and $\{v_{\delta(g)}\}_g$. Thus $[bq'b^*, u_g] = 0$, $[q'b^*bq', v_{\delta(g)}] = 0$, $\forall g \in \Gamma$. Since u_g, v_h normalize B (resp. A^n), this implies that for all $g \in \Gamma$ we have

$$[\Sigma_h q_h \theta_h(q'), u_g] = [E_B(bb^*), u_g] = 0,$$

$$[\Sigma_h \theta_{h^{-1}}(q_h) q', v_{\delta(g)}] = [E_B(b^*b), v_{\delta(g)}] = 0,$$

in the embedding case, and

$$[\Sigma_h q_h \theta_h(q'), u_g] = [E_{A^n}(bb^*), u_g] = 0,$$
$$[\Sigma_h \theta_{h^{-1}}(q_h) q', v_{\delta(g)}] = [E_{A^n}(b^* b), v_{\delta(g)}] = 0,$$

in the local OE case, where we have still denoted (and will do so hereafter) by θ_h the automorphism $\mathrm{Ad}(v_h) = \theta'_h$ on A^n, for simplicity.

Applying this to non-zero spectral projections q' of $\Sigma_h \theta_{h^{-1}}(q_h)$, by the ergodicity of $\sigma_g = \mathrm{Ad}(u_g), g \in \Gamma$, on A it follows that $\Sigma_h q_h \theta_h(q')$ is equal to $p = \chi_X$. Thus $q_h \leq \theta_h(q')$, or equivalently $\theta_{h^{-1}}(q_h) \leq q'$, for all $h \in \Lambda$. Summing up over $h \in \Lambda$ it follows that the support projection q of $\Sigma_h \theta_{h^{-1}}(q_h)$ is majorized by q'. This shows $b^* b = \Sigma_h \theta_{h^{-1}}(q_h) = cq$ for some scalar c, necessarily an integer with $c = \tau(p)/\tau(q)$. Thus, $c^{-1/2} b$ is a partial isometry with right support $q \in A^n$. On the other hand, taking in the above q' to be a projection in Bq (resp. in $A^n q$) fixed by $\{v_{\delta(g)} \mid g \in \Gamma\}$, this also shows that $q' = q$, i.e. $\{v_{\delta(g)}\}_g$ must act ergodically on Bq (resp. $A^n q$).

Let us first finalize the proof of case Δ is a local OE, i.e. when $M^n \supset P$, $A^n \supset B$. In this case, σ weak mixing on $L^\infty X = A = A^n p$ implies that $\{u_g\}'_g \cap M$ is contained in the center of the group von Neumann algebra $L\Gamma = \{u_g\}''_g \subset M$, so in particular the projection $e = c^{-1} bb^*$ lies in $\mathcal{Z}(L\Gamma)$. Thus, $u_g e \mapsto v_{\delta(g)} q$ is an equivalence between the left regular representations of $\Gamma/ker(\delta)$ and $\delta(\Gamma) = \Lambda_0$ (with respect to the identification of these two groups implemented by δ), spatially implemented by $c^{-1/2} b$.

Noticing that all the coefficients of $bb^* = \Sigma_h(\Sigma_l q_{hl} \theta_h(q_l)) v_h$ are projections, it follows that if we denote $K = \ker(\delta)$ then $c = |K|$, $b = \Sigma_{k \in K} u_k$ and $e = |K|^{-1} \Sigma_{k \in K} u_k$. Moreover, since $qM^n q = A^n q \vee \{v_h q \mid h \in \Lambda_0\}$, we have $q v_h q = 0, \forall h \in \Lambda \setminus \Lambda_0$ implying that $\Lambda_0 = \delta(\Gamma)$ has index $[\Lambda : \Lambda_0] = n = \tau(q)^{-1}$.

Assume now that Δ is an embedding, i.e. $M^n \subset P$, $A^n = B$. Let $z = E_M(bb^*) \in M$ and s the support projection of z. Since bb^* commutes with $\{u_g\}_g$, it follows that $z \in L\Gamma' \cap M = \mathcal{Z}(L\Gamma)$ (because σ is weak mixing on A). Since $\{v_h \mid h \in \Lambda_0\}$ is in its standard representation on $q(L^2(L\Lambda_0)) \simeq \ell^2 \Lambda_0$ and the isomorphism $u_g s \mapsto v_{\delta(g)} q$ is spatially implemented, it follows that z must be a multiple of s which in turn must equal a minimal projection in the algebra generated by $u_g, g \in K = ker(\delta)$, lying in the center of this algebra. On the other hand, since $bb^* = \Sigma_h(\Sigma_l q_{hl} \theta_h(q_l)) v_h$ and $\{\theta_h(q_l)\}_l$ are mutually orthogonal, it follows that the Fourier coefficients a_h of $bb^* = \Sigma_h a_h v_h$ are projections. Using that $\{p_h^g\}_h$ are mutually orthogonal, this implies the Fourier coefficients of $z = E_M(bb^*)$ in $\{u_g\}_g$ are projections as well. This shows that we must have $z = \Sigma_{k \in K} u_k$.

On the other hand, by the form of b we have $Ab \supset bA$ and $b^* A b = Aq$, thus $bb^* A bb^* \subset A bb^*$ implying that the support projection e of bb^* implements a τ_μ-preserving conditional expectation E_0 of A onto a subalgebra $A_0 \subset A$ by $E_0(a) = \tau_M(e)^{-1} E_A(eae), a \in A$. Thus, if we denote by $\mathcal{K} \subset \mathcal{R}_\theta^{Y_0}$ the sub-equivalence relation

implemented by the partial isometries $a_h v_h$ then $\mathcal{K}$ contains $\mathcal{R}_{\sigma(K)}$, $A_0 = L^\infty(X/\mathcal{K})$, all orbits of $\mathcal{K}$ have m points (μ-a.e.), $m = \tau_M(e)^{-1}$ and by the definitions α intertwines the restriction of the Γ/K-action implemented by σ on $A_0 = A^{\mathcal{K}}$ with the $\Lambda_0 = \delta(\Gamma)$ action implemented by $\theta_{|\Lambda_0}$ on $A^n q = Bq$. The last part of the statement is now trivial. $\square$

6. Final remarks

6.1. Proposition 5.11 shows that all OE rigidity results 5.6-5.8 hold in fact true without the assumption that the source group Γ has no finite normal subgroups, provided we replace "conjugate" by "virtual conjugate" (in the sense of [Fu1,2]) in the conclusions. Moreover, if we are only interested in getting a "virtual conjugacy" conclusion, then by 5.11 we do not need the assumption *any* Γ*-invariant finite subequivalence relation of* $\mathcal{R}^t_\theta$ *is contained in* $\mathcal{R}_\sigma$ in the statement of Superrigidity of Embeddings 5.8. On the other hand, note that in order to get a "conjugacy" conclusion in 5.8, the above condition is unavoidable, as shown by the following example: Let $\Lambda = \Gamma \times K_0$, with K_0 a finite group, and $\Lambda \curvearrowright^{\theta'} X$ a free action. Let $\Lambda \curvearrowright^{\theta} (X/K_0 \times K_0)$ be the product of the action $\Gamma \curvearrowright X/K_0$ implemented by θ' and the action of K_0 on itself by left translation. It is trivial to see that θ, θ' are orbit equivalent. If $\sigma = \theta'_{|\Gamma}$ then the inclusion $\Gamma \subset \Gamma \times K_0 = \Lambda$ implements an emebedding of equivalence relation $\mathcal{R}_\sigma \subset \mathcal{R}_{\theta'}$. Thus $\mathcal{R}_\sigma \subset \mathcal{R}_\theta$. However, if Γ is w-rigid, $K_0 \neq 1$ and σ is say a Bernoulli Γ-action, then it is easy to see that σ cannot be conjugate to the restriction of θ to a subgroup Λ_0 of $\Gamma \times K_0$.

6.2. One can obtain a cocycle superrigidity result similar to 5.2 in which all assumptions are the same except that "s-malleable" is replaced by the weaker assumption "malleable" (as defined in the first part of 4.3), but where the target group is in the smaller class of Polish groups of *finite type I*, $\mathscr{U}_{I,fin}$, i.e. Polish groups that can be embedded as closed subgroups of the unitary groups of finite type I von Neumann algebras (e.g. compact groups, or residually finite discrete groups). Moreover, the same is true if σ is merely *sub-malleable*, i.e. if it can be extended to a malleable action with the relative weak mixing condition satisfied. The proof of this fact is simpler than that of Theorem 5.2, closely mimicking proofs in ([P1], [PSa]).

6.3. Alex Furman noticed in ([Fu4]) that Gaussian actions (see e.g. [CCJJV] for the definition) are also s-malleable. The proof is the same as the proof of s-malleability of their non-commutative version, the Bogoliubov actions in (1.6.3 of [P2]). Gaussian actions are weak mixing whenever they come from orthogonal representations with no finite dimensional invariant subspaces. It would be interesting to produce more examples of malleable actions, or more generally of quotients of malleable actions. It seems to us that the action obtained by taking a group embedding $\Gamma \subset \mathcal{G}$ into a compact (Lie) group $\mathcal{G}$ with $\overline{\Gamma} = \mathcal{G}$ and letting Γ act on $\mathcal{G}$ with its Haar measure by

(left) translation (page 1085 in [Fu2]) should be a (quotient of) malleable action. But being compact, these actions are not weak mixing.

6.4. As already pointed out in ([P4]), if σ is a Bernoulli action of an infinite Kazhdan group Γ on the probability space (X, μ) and $K \subset \mathrm{Aut}(X, \mu)$ is a finite abelian group commuting with $\sigma(\Gamma)$ then the "quotient action" σ^K of Γ on X/K implemented by σ is free mixing but not sub malleable, in the sense of 6.2 above. Indeed, by ([P4]) we have $\mathrm{H}^1(\sigma^K) = \mathrm{Hom}(\Gamma \times K, \mathbb{T})$ while if σ^K would be sub malleable then by ([P4], [PSa]) we would have $\mathrm{H}^1(\sigma^K) = \mathrm{Hom}(\Gamma, \mathbb{T})$. This is a contradiction, since for K non-trivial $\mathrm{Hom}(\Gamma \times K, \mathbb{T}) = \mathrm{Hom}(\Gamma, \mathbb{T}) \times \hat{K}$ has more elements than the (finite) group $\mathrm{Hom}(\Gamma, \mathbb{T})$.

Another class of actions that are not sub malleable are the *rigid* actions, as defined in (5.10.1 of [P5]). The prototype such action is obtained by taking a rigid inclusion of groups of the form $\mathbb{Z}^n \subset \mathbb{Z}^n \rtimes \Gamma$ (e.g. $n = 2$ and $\Gamma \subset SL(2, \mathbb{Z})$ non-amenable), and letting σ be the action of Γ on $\mathbb{T}^n = \hat{\mathbb{Z}^n}$ induced by the action of Γ on $\mathbb{Z}^n$. In fact, it it can be easily shown using same arguments as in (Sec. 6 of [P3]) that $\Gamma \curvearrowright \mathbb{T}^n$ cannot be realized as a quotient of a malleable action.

6.5. It would be interesting to have an abstract characterization of the Polish groups of finite type. Note in this respect that any such group $\mathcal{V}$ is isomorphic to a close subgroup of the unitary group of a (separable) Hilbert space and admits a complete metric which is both left and right invariant. Are these conditions sufficient to insure that $\mathcal{V} \in \mathscr{U}_{fin}$? As far as "classic" groups are concerned, it is interesting to recall an old result of Kadison and Singer ([KaSi]; see also [D1]), improving on an earlier result of von Neumann and Segal ([vNS]), showing that if a connected locally compact group G can be faithfully represented into the unitary group of a finite von Neumann algebra then $G = K \times H$ where K is connected compact and H is a vector group (I am grateful to Dick Kadison and Raja Varadarajan for pointing out to me this result). Thus, by Proposition 3.5, any Polish group that contains the homeomorphic image of a connected locally compact group G which is not of this form is not in the class $\mathscr{U}_{fin}$ either. Thus, the class $\mathcal{U}_{fin}$ of "target groups" in our cocycle superrigidity results 5.2/5.3 is essentially disjoint from linear algebraic groups, which are the target groups in Zimmer's cocycle superrigidity ([Z1,2]). As pointed out by Alex Furman, the fact that $G = \mathrm{GL}(n)$ and other linear algebraic groups are not in $\mathscr{U}_{fin}$ follows also as a combination of Zimmer's result and the "hereditary principle" 3.5.

6.6. In its general form, the cocycle superrigidity result 5.2 shows that any cocycle for a diagonal product Γ-action $\sigma \times \rho$, with Γ baring some mild *rigidity* (of Kazhdan-type) and σ some *malleability* property, is "absorbed" by the ρ action. The validity of such "principle" to situations more general than the ones covered by 5.2 may lead to further applications, notably to the calculation of the Feldman-Moore higher cohomology groups $\mathrm{H}^n(\mathcal{R}_\sigma), n \geq 2$, of such Γ-actions σ (see [FM] for the definition of H^n). In this respect, we expect that group actions satisfying condition 5.6.1 should satisfy

$\mathrm{H}^2(\mathcal{R}_\sigma) = \mathrm{H}^2(\Gamma)$ ($\mathbb{T}$-valued 2-cocycles for Γ). In this same vein, it would be of great interest to extend the class of target groups $\mathcal{V}$ covered by the Cocycle Superrigidity 5.2, 5.3 from $\mathscr{U}_{fin}$ to a larger class. It seems reasonable to believe that such larger class may include all Polish subgroups of the unitary group $\mathcal{U}(\mathcal{H})$ on Hilbert space, in particular all separable locally compact groups. In other words, the following may hold true: *If Γ has an infinite w-normal rigid subgroup H and $\Gamma \curvearrowright X$ is s-malleable and weak mixing on H then any $\mathcal{V}$-valued cocycle for σ, with $\mathcal{V} \subset \mathcal{U}(\mathcal{H})$ a closed subgroup, is cohomologous to a group morphism of Γ into $\mathcal{V}$.*

6.7. It would be extremely interesting to characterize the class $\mathcal{CS}$ of groups Γ for which any s-malleable weak mixing Γ-action $\Gamma \curvearrowright^{\sigma} X$ (e.g. $\Gamma \curvearrowright [0,1]^\Gamma$) is $\mathcal{U}_{fin}$-cocycle superrigid, i.e. for which any measurable $\mathcal{V}$-valued cocycle over σ is cohomologous to a group morphism of Γ into $\mathcal{V}$, for any $\mathcal{V} \in \mathcal{U}_{fin}$. The class $\mathcal{CS}$ should be much larger than the class of groups considered in this paper. It cannot, of course, contain the free groups (see e.g. Sec. 3 in [P4]), but may contain all non-amenable groups of the form $\Gamma = \Gamma_0 \times \Gamma_1$ with both Γ_i infinite, thus relating with the rigidity results of Monod-Shalom ([MoSh1,2]) and Hjorth-Kechris ([HjKe]).

6.8. Let $\{\Gamma_i\}_{i\in I}$ be a countable (at most) family of groups such that each Γ_i has a wq-normal rigid subgroup $H_i \subset \Gamma_i$. Denote $\Gamma = *_i\Gamma_i$ the free product of the groups Γ_i. As pointed out in ([P4]), if $|I| \geq 2$ then Γ does not have any wq-normal rigid subgroup. Nevertheless, the following version of cocycle superrigidity does hold true for these free product groups: Let σ be an action of Γ on the standard probability space (X, μ) and $\mathcal{V} \in \mathscr{U}_{fin}$. Assume that for each $i \in I$, $\sigma_{|\Gamma_i}$ is s-malleable and either mixing or weak mixing with H_i w-normal in Γ_i. If $w : X \times \Gamma \to \mathcal{V}$ is a measurable cocycle for σ then there exist group morphisms $\delta_i : \Gamma_i \to \mathcal{V}$ and measurable maps $v_i : X \to \mathcal{V}$ such that $w_g = v_i^*\gamma_i(g)\sigma_g(v_i)$, $\forall g \in \Gamma_i$, $\forall i \in I$. When combined with Proposition 5.11, this can be used to obtain an OE rigidity result of Bass-Serre type, in the spirit of results in ([IPeP]).

6.9. If one restricts the proofs in Sec. 2-4, leading to the proof of the Cocycle Superrigidity 5.2/5.3, to the case the groups act on commutative algebras, then the arguments can be easily translated into measure theoretical terms. We opted for a von Neumann algebra presentation because the ideas behind the proofs are so much "von Neumann algebra" in spirit and because of the non-commutative generalizations it allows (e.g. 5.5). Another reason was that we needed this setting for Proposition 5.11, which we proved using von Neumann algebra analysis. In this respect, we mention that shortly after the initial version of this paper was circulated, Stefaan Vaes in ([V]) and then Alex Furman ([Fu4]) were able to give alternative, genuine measure theory proof to the "OE case" of 5.11, using 3.3 in [Fu2]. Their expository notes also contain measure theoretical presentations of our proof of 5.2.

On the other hand, note that we only used the Cocycle Superrigidity 5.2 towards

establishing rigidity results for embeddings and local OE of actions. It should be possible to derive from 5.2 rigidity results for arbitrary morphisms (as defined in 1.4.2) between equivalence relations implemented by free m.p. actions. Such morphisms still give rise to cocycles, which by 5.2 can be untwisted whenever the source action satisfies 5.6.1. But it is not clear how to interpret the "untwister" as some kind of "generalized conjugator" in this generality. In fact, one needs to first understand what rigidity conclusion one seeks to derive for arbitrary morphisms between these equivalence relations. The following type of morphisms, generalizing both local OE and embeddings, could be easier to analize: If $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are free ergodic m.p. actions, then a morphism $\Delta : (X, \mu) \to (Y, \nu)$ between $\mathcal{R}_\sigma, \mathcal{R}_\theta$ is a *local embedding* if it is 1 to 1 (but not necessarily onto) on μ-almost every Γ-orbit. The "ideal" superrigidity statement for a local embedding Δ should generalize both 5.7, 5.8. It should (roughly) show that there exist $\Lambda_0 \subset \Lambda$ and $\alpha \in [\Gamma]$, $\beta \in [\Lambda]$ such that the Δ-pull back of $\beta\Lambda_0\beta^{-1}$ is conjugate to a quotient of $\alpha\Gamma\alpha^{-1}$.

6.10. Several applications of the Cocycle Superrigidity 5.2/5.3 have been obtained since the initial circulation of this paper in Dec. 2005. This includes Theorem 5.7 (solving 5.7.2° in [P2]) and Corollary 5.10 in the present version of the paper, and a result by Stefaan Vaes and the author showing that if $\Gamma \curvearrowright X$ satisfies 5.6.1, $K \curvearrowright X$ is a compact action commuting with it and so that $\Gamma \curvearrowright X/K$ is still free, then $\Gamma \curvearrowright X/K$ is OE superrigid ([PV]). Also, Alex Furman used 5.2 to show that if Γ, Λ are lattices in a higher rank semisimple Lie group $\mathcal{G}$ then the action $\Gamma \curvearrowright \mathcal{G}/\Lambda$ cannot be realized as a quotient of a Bernoulli Γ-action, more generally of a s-malleable weak mixing action ([Fu4]). This solves a problem posed in ([P3], see comments after 7.7) and shows that the case Γ higher rank lattice of the OE Superrigidity 5.6 is already covered by the OE Superrigidity in ([Fu2]). Finally, Simon Thomas in ([T]) used the Cocycle Superrigidity 5.2 to answer some open problems in descriptive set theory (Borel equivalence relations), showing for instance that the universal countable Borel equivalence relation E_∞ cannot be implemented by a free action of a countable group (see [JaKeL] for definitions).

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